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Finite Dimensionality and Upper Semicontinuity of the Global Attractor of Singularly Perturbed Hodgkin–Huxley Systems

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In an effort to show that the standard Hodgkin–Huxley system approximates the hyperbolic Hodgkin–Huxley system we view the hyperbolic system as a singular perturbation of the standard system. We establish the existence of global attractors for the singularly perturbed hyperbolic system and show that they have finite fractal and Hausdorff dimensions. We then show that the global attractors of the singularly perturbed system converge to the global attractor for the standard system as the small perturbation parameter decreases to zero. © 1996 Academic Press, Inc.

1. INTRODUCTION

In their celebrated work of the early fifties A. L. Hodgkin and A. F. Huxley developed a system of equations to model the excitation of the giant axon of the squid Loligo, cf. [26–29]. This system still provides a suitable basis for describing the macroscopic ionic current of the giant axon of the squid. In addition it has been useful for qualitatively describing excitation phenomena for macroreceptors and other natural membranes. In formulating their model Hodgkin and Huxley visualized the axon as a cable consisting of a neural core surrounded by a membrane across which currents are allowed to travel back and forth through both capacitive and ion transport mechanisms.

The standard Hodgkin–Huxley model is given by the following distributed parameter system which couples a parabolic equation with spatially dependent ordinary differential equations:

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$$\partial V/\partial t - \partial^2 V/\partial x^2 = g_{N_a} m^3 h(V_{N_a} - V) + g_K n^4 (V_K - V) + g_L (V_L - V) \tag{1.1a}$$

$$\partial m/\partial t = (m_\infty(V) - m)/\tau_m \tag{1.1b}$$

$$\partial h/\partial t = (h_\infty(V) - h)/\tau_h \tag{1.1c}$$

$$\partial n/\partial t = (n_\infty(V) - n)/\tau_n \tag{1.1d}$$

Here x represents the longitudinal displacement along the axon and t is time; V is the electrical potential in the nerve; m, h, n are chemical concentrations of sodium N_a, potassium K, and other (leakage) ions which are nonnegative; g_{N_a}, g_K, g_L are the maximum conductances of these ions; V_{N_a}, V_K, V_L are equilibrium potentials for these ions; m_∞, h_∞, n_∞ are steady state values and τ_m, τ_h, τ_n are relaxation times. A thorough discussion of these terms is given by Cronin [6], also cf. [16].

The leading equation (1.1a) is intended as an approximation of the cable or telegrapher's equation rather than a diffusion equation. It is therefore reasonable to replace (1.1a) by

$$\begin{aligned} &\varepsilon\, \partial^2 V/\partial t^2 + (\varepsilon f(m, h) + 1)\, \partial V/\partial t - \partial^2 V/\partial x^2 \\ &\quad = g_{N_a} m^3 h(V_{N_a} - V) + g_K n^4 (V_K - V) + g_L (V_L - V) \end{aligned} \tag{1.1a$'$}$$

where $\varepsilon > 0$ is a small positive parameter representing inductance in the system. Physical reasoning asserts that inductance in the system is small and therefore terms involving ε can be effectively ignored. Numerical support for these physical heuristics are provided by Lieberstein [32]. This work suggests that the solutions of (1.1a′), (1.1b–d) converge to solutions of (1.1a–d) as ε decreases to zero. These conjectures are rigorously established in [17], also cf. [18].

In the work at hand we continue with the ideas initiated in [18]. In particular, it is known that within certain parameter ranges, cf. Golubitsky and Schaeffer [21], the global dynamics of the Hodgkin–Huxley system possess a rich qualitative structure. It thus becomes reasonable to examine the global dynamics of the singularly perturbed system and to ascertain whether or not they approximate those of the standard system for small values of ε. Toward this end we establish the existence of global attractors for (1.1a′), (1.1b–d) for sufficiently small $\varepsilon > 0$, show that these attractors have finite fractal and Hausdorff dimensions, and describe their convergence as $\varepsilon \downarrow 0$.

We point out that Hale and Raugel have undertaken a similar analysis for a single hyperbolic singular perturbation of a semilinear parabolic equation, cf. [23, 24]. The hyperbolic singular perturbation of a simplified neural conduction model, the FitzHugh–Nagumo system, is treated by Valencia [45]. Additional treatments of hyperbolic singular perturbations (and the convergence thereof) are given by Najman [38], Benaouda and

Tort [4], Vishik and Lyusternik [47], Mora and Sola–Morales [37], Smoller [43], and [17, 18].

We shall draw heavily upon the theory of semigroups, attractors and invariant sets for systems of partial differential equations. For general references on semigroups and their attractors we refer the reader to Goldstein [19], Fattorini [14], Pazy [39], Hale [22], Vishik [46], Babin and Vishik [2], Martin [34], Temam [44], and Ladyshenkaya [31]. Results on invariant sets for partial differential equations are found in Cheuh, Conley and Smoller [7], Smoller [42], and Bates and Jones [3].

The fundamental analytical work for system (1.1a–d) was done by J. Evans, cf. [9–12] and Evans and Shenk [13]. Rauch and Smoller [40] considered the related FitzHugh–Nagumo System and Mascagni [35] repeats the work of [13] within the context of a bounded domain with homogenous Neumann boundary conditions. Marion [33] adapted invariant set techniques to partially dissipative reaction diffusion systems and computed finite Hausdorff and fractal dimensions for the attractor of (1.1a–d). Other references on Hodgkin–Huxley equations include Meves [36] and Awiszus, Dehnhardt, and Funke [1].

2. Preliminaries on Function Spaces, Flows, and Attractors

In what follows we shall work in a variety of function spaces defined on the interval $(0, 1)$. The norm of the standard Hilbert space $L_2(0, 1)$ will be denoted by $\| \ \|$ and the inner product will be given by $\langle \ \rangle$. When we use the inner product in other Hilbert spaces we shall subscript the brackets. The norms of the Hilbert Sobolev spaces $H^r(0, 1)$ will be given by $\| \ \|^{(r)}$ and the norm of $L_\infty(0, 1)$ will be denoted by $\| \ \|_\infty$. We shall frequently work in Hilbert spaces which are products of Hilbert spaces. In particular we shall define the spaces $X_1 = H^1(0, 1) \times L_2(0, 1)$, $X_2 = H^2(0, 1) \times H^1(0, 1)$, $X_3 = H^3(0, 1) \times H^2(0, 1)$, $Y_1 = H^1(0, 1) \times L_2(0, 1) \times L_2^+(0, 1)$, $Y_2 = H^2(0, 1) \times H^1(0, 1) \times H^1_+(0, 1)$ and $Y_3 = H^3(0, 1) \times H^2(0, 1) \times H^2_+(0, 1)$.

The appropriate norms are given by

$$\|(u, v)^T\|_{X_1} = ((\|u\|^{(1)})^2 + \|v\|^2)^{1/2} \tag{2.1a}$$

$$\|(u, v)^T\|_{X_2} = ((\|u\|^{(2)})^2 + (\|v\|^{(1)})^2)^{1/2} \tag{2.1b}$$

$$\|(u, v)^T\|_{X_3} = ((\|u\|^{(3)})^2 + (\|v\|^{(2)})^2)^{1/2} \tag{2.1c}$$

$$\|(u, v, w)^T\|_{Y_1} = ((\|u\|^{(1)})^2 + \|v\|^2 + \|w\|^2)^{1/2} \tag{2.1d}$$

$$\|(u, v, w)^T\|_{Y_2} = ((\|u\|^{(2)})^2 + (\|v\|^{(1)})^2 + (\|w\|^{(1)})^2)^{1/2} \tag{2.1e}$$

$$\|(u, v, w)^T\|_{Y_3} = ((\|u\|^{(3)})^2 + (\|v\|^{(2)})^2 + (\|w\|^{(2)})^2)^{1/2}. \tag{2.1f}$$

We shall assume some familiarity with the basic theory of abstract dynamical systems and semigroups. If Y is a general Banach space and $\sigma(\cdot, \cdot)\colon \mathbb{R}^+ \times Y \to Y$ is a semiflow, then we recall that B is said to be an absorbing set for the semiflow if, for every bounded subset $Z \subset Y$, there exists a T (which may depend on Z) so that for all $z \in Z$ and $t \geqslant T$, we have $\sigma(t, z) \in B$. We define the semidistance between two sets B_1 and B_2 in a Banach space Y by

$$\delta_Y(B_1, B_2) = \sup_{b_1 \in B_1} \operatorname{dist}_Y(b_1, B_2) = \sup_{b_1 \in B_1} \inf_{b_2 \in B_2} \|b_1 - b_2\|_Y. \tag{2.2}$$

We shall find it convenient to work within the context of one parameter semigroups of operators and we recall that the semigroup $\{S(t) \mid t \geqslant 0\}$ naturally associated with a semiflow σ is defined by $S(t)\, y = \sigma(t, y)$ for $t \geqslant 0$ and $y \in Y$. A compact set $\mathscr{A} \subset Y$ is called a global attractor for a flow if the following two properties hold

$$S(t)\mathscr{A} = \mathscr{A} \qquad \text{for any} \quad t \geqslant 0 \tag{2.3a}$$

$$\lim_{t \to \infty} \delta_Y(S(t) B, \mathscr{A}) = 0 \qquad \text{for any bounded subset } B \subset Y. \tag{2.3b}$$

The following result of Temam [44, p. 23] provides conditions sufficient to guarantee the existence of a global attractor.

THEOREM 2.4. *Let $\{S(t) \mid t \geqslant 0\}$ be a one parameter semigroup of operators on a Banach X, which admits a decomposition $S(t) = S_1(t) + S_2(t)$ into one parameter families of operators $\{S_1(t) \mid t \geqslant 0\}$ and $\{S_2(t) \mid t \geqslant 0\}$ that satisfy the following conditions*:

$$\text{for every bounded subset } Z \subset Y \ \lim_{t \to \infty} \sup\{\|S_1(t) v\|_Y : v \in Z\} = 0, \tag{2.5a}$$

$S_2(t)$ is uniformly compact for large t, i.e., for every bounded subset

$$Z \subset Y \text{ there is a } t_0 > 0 \text{ so that } \bigcup_{t \geqslant t_0} S_2(t)\, Z \text{ is precompact in } Y. \tag{2.5b}$$

Then there exists a global attractor $\mathscr{A}$ for the semiflow associated with $\{S(t) \mid t \geqslant 0\}$ provided that there exists a bounded absorbing set $B \subset Y$. Moreover $\mathscr{A} = \omega(B)$ where $\omega(B)$ is the omega limit set of B.

3. THE PARABOLIC SYSTEM

In this section we specify the form of the Hodgkin–Huxley system to be considered. Our work will rely upon a priori estimates and general results on dissipative dynamical systems and not upon specific calculation of parameter dependent qualitative dynamics. We therefore find it convenient to consider a reduced Hodgkin–Huxley system which incorporates the salient features of the original system. Following the ideas of Evans and Shenk [13] our simplification couples a parabolic equation with a single spatially dependent ordinary differential equation. We point out that the subsequent development could have been carried out for the full four component system. However, this would have only served to further obfuscate a treatment which is already computationally cumbersome. Therefore we consider the two component system:

$$u_t - u_{xx} = -f_1(w)\, u + f_2(w) \tag{3.1a}$$

$$w_t = -h_1(u)\, w + h_2(u), \qquad t > 0, \quad x \in (0, 1). \tag{3.1b}$$

Normally one would expect a positive constant appearing as the coefficient of u_t but we will lose no generality by assuming that it is unity.

We shall assume that the axon has finite length and we need to impose boundary conditions. We augment the leading equation with the homogeneous Neumann boundary conditions

$$u_x(0, t) = u_x(1, t) = 0, \qquad t \geqslant 0. \tag{3.2}$$

If we wished to consider an experimental situation where a variable current was applied to one of the ends (say the left end) we would need a boundary condition of the form

$$u_x(0, t) = ri(t). \tag{3.3}$$

However, we will not be considering this case.

Cronin [6] provides a careful description of the physiological kinetics of the Hodgkin–Huxley system and the reader is referred thereto to ascertain that the following assumptions we make on our nonlinearites are physically reasonable. We make the following assumptions:

(i) there exists $a > 0$ so that $f_1(w) \geqslant a$ for all w,

(ii) there exists $b > 0$ so that $h_1(u) \geqslant b$ for all u,

(iii) $h_2(u) \geqslant 0$ for all u,

(iv) $f_1(\cdot), f_2(\cdot)$ are smooth nonnegative functions; they and their derivatives are polynomially bounded,

(v) $h_1(\cdot)$ and $h_2(\cdot)$ are smooth and uniformly bounded functions with locally bounded derivatives.

Finally we need to prescribe initial conditions:

$$u(x, 0) = u_0(x) \tag{3.4a}$$

$$w(x, 0) = w_0(x) \geqslant 0 \qquad \text{for} \quad x \in (0, 1). \tag{3.4b}$$

Our system could be called a partially diffusive reactive system or in the terminology of M. Marion [33], a partially dissipative reaction diffusion system. The natural function space for the analysis of such systems is probably a continuous function space. However the natural spaces for analyzing attractors and for studying the singularly perturbed hyperbolic system are Hilbert Sobolev spaces. We observe that because (3.1b) is a spatially dependent ordinary differential equation we expect no regularization of discontinuous or rough initial data, w_0, and we therefore need to work with strong solutions. We shall assume that the reader is sufficiently familiar with semigroup theory to be comfortable with the notion of strong solutions, cf. Pazy [39], Goldstein [19]. We have the following global existence theorem.

THEOREM 3.5. *If $u_0(\cdot) \in H^1(0, 1)$ and $w_0(\cdot) \in L_\infty(0, 1)$ $(w_0 \geqslant 0)$ then there exists a unique globally defined strong solution pair $(u(\cdot, \cdot), w(\cdot, \cdot))$ of* (3.1–3.2, 3.4) *on* $(0, 1) \times [0, \infty)$. *There exists a constant $C > 0$ which depends on $\|w_0\|_\infty$ and $\|u_0\|^{(1)}$ so that*

$$\sup_{t \geqslant 0} \{ \|u(\cdot, t)\|^{(1)}, \|w(\cdot, t)\|_\infty \} \leqslant C. \tag{3.6}$$

Moreover $w(x, t) \geqslant 0$ for $x \in (0, 1)$, $t > 0$.

Outline of Proof. The crux of the proof is finding uniform estimates. If we formally integrate (3.1b) we observe that

$$w(x, t) = \exp\left(-\int_0^t h_1(u(s, x))\, ds \right) w_0(x) + \int_0^t h_2(u(s, x)) \times \exp\left(\int_s^t h_1(u(r, x))\, dr \right) ds. \tag{3.7}$$

Consequently there exists a constant C_1 which depends on h_2 and b so that

$$0 \leqslant w(x, t) \leqslant \|w_0\|_\infty + C_1 \equiv C_w. \tag{3.8}$$

We therefore are assured a $C_f > 0$ so that

$$\sup_{t > 0} \{ \|f_i(w(x, t))\|_\infty, i = 1 \text{ or } 2\} \leqslant C_f. \tag{3.9}$$

Because $f_1(w(x, t)) \geqslant a$ it is possible to find an interval $[-k, k]$ containing $u_0(x)$ for $x \in (0, 1)$ so that if $0 \leqslant w \leqslant C_w$

$$-f_1(w)(-k) + f_2(w) \geqslant 0 \tag{3.10a}$$

and

$$-f_1(w)(k) + f_2(w) \leqslant 0. \tag{3.10b}$$

We have thereby constructed an invariant rectangle (in this case interval) for solutions to (3.1a) and solutions must satisfy

$$-k \leqslant u(x, t) \leqslant k,$$

cf. Cheuh, Conley and Smoller [7] or Smoller [42]. In the presence of uniform estimates, standard fixed point and semigroup continuation arguments yield globally defined solutions, cf. Martin [34].

If $g(\cdot, \cdot)$ is the heat kernel associated with $z_{xx} - z_t = 0$ and homogeneous Neumann boundary conditions then it is seen that solutions to (3.1a) can be represented as

$$\begin{aligned} u(x, t) = \int_0^1 g(x - \zeta, t)\, u_0(\zeta)\, d\zeta + \int_0^t \int_0^1 g(x - \zeta, t - s) \\ \times \{-f_1(w(\zeta, s))\, u(\zeta, s) + f_2(w(\zeta, s))\}\, d\zeta\, ds. \end{aligned} \tag{3.11}$$

If we define a nonlinear mapping $F(\cdot, \cdot)$ by

$$F(u, w) = -f_1(w)\, u + f_2(w) \tag{3.12}$$

and let $\{T(t) \mid t \geqslant 0\}$ be the analytic semigroup of nonexpansive operators on $L_2(0, 1)$ with the infinitesimal generator $-A$ defined as

$$(Au)(x) = -u''(x) \tag{3.13}$$

with

$$D(A) = \{u \mid u \in H^2(0, 1) \text{ with } u_x(0, t) = u_x(1, t) = 0\}, \tag{3.14}$$

then (3.11) has semigroup formulation

$$u(\cdot, t) = T(t)\, u_0 + \int_0^t T(t - s)\, F(u(\cdot, s)\, w(\cdot, s))\, ds \tag{3.15}$$

If A is the operator defined by (3.13) and (3.14) we observe that because A is nonnegative and self-adjoint, fractional powers of A can be defined via the spectral calculus. If we impose the graph norm on $D(A^{1/2})$ we may observe that

$$\|u\|_{D(A^{1/2})} = \|u\|^{(1)}.$$

Furthermore this norm can be shown to be equivalent to $\|(I+A)^{1/2} u\|$.

Moreover, we may associate a semiflow on X_1 with solution pairs $(u(\cdot, t), w(\cdot, t))$. Indeed we may define a semigroup $\{S(t) \mid t \geqslant 0\}$ by

$$S(t)(u_0, w_0) = (u(\cdot, t), w(\cdot, t)). \tag{3.16}$$

It is possible to produce uniform estimates for $\|u_x(\cdot, t)\|$ and $\|w_t(\cdot, t)\|_\infty$. If we assume that $u_0 \in H^2(0, 1)$ and $w_0 \in H^1(0, 1)$ then we can produce uniform estimates for $\|u_t(\cdot, t)\|$, $\|u_{tt}(\cdot, t)\|$, $\|u_{xx}(\cdot, t)\|$, $\|w(\cdot, t)\|^{(1)}$ and $\|w_t(\cdot, t)\|^{(1)}$. It is then possible to show that the semigroup defined by (3.16) will produce semiflows on the Banach spaces X_1 and X_2. In fact, we make minor adjustments in the invariant region argument of Marion [33] to show that the semigroup $S(t)$ given by (3.16) will define a flow in the larger space $L_2(\Omega) \times L_2(\Omega)$ possessing a global attractor $\mathscr{A}$ which has finite Hausdorff and fractal dimensions. When we consider the singularity perturbed problem we shall need to work in spaces with more regularity. One approach would be to apply regularity arguments to show that $\mathscr{A}$ lies in these spaces also. However, we shall proceed on a different track. The arguments which we use to produce the regularity of the global attractor for the singularly perturbed Hodgkin–Huxley system are based on energy type estimates. These methods immediately apply to producing the same estimates for the parabolic system. Therefore arguments appearing in subsequent sections will yield the following theorem.

THEOREM 3.17. *If $(u_0, w_0) \in X_2 = H^2(0, 1) \times H^1(0, 1)$ then the semiflow associated with the semigroup $\{S(t) \mid t \geqslant 0\}$ has a global attractor $\mathscr{A}$ in X_2. The attractor $\mathscr{A}$ has finite Hausdorff and fractal dimensions.*

Our eventual goal is to demonstrate the convergence of global attractors for the singularly perturbed Hodgkin–Huxley system to the attractor given above. The leading equation for the singularly perturbed Hodgkin–Huxley system is the cable or telegrapher's equation and therefore, because we are working with a second order hyperbolic equation, the solution semigroup must account for the time derivative, u_t, of the state variable as well as the state variable u. Therefore we need to embed the semiflow defined by $(u(\cdot, t), w(\cdot, t)) = S(t)(u_0, w_0)$ into Y_1, and the attractor $\mathscr{A}$ into a three component function space which will account for u_t as well as u.

If we differentiate (3.1a) with respect to t we obtain the three component system

$$u_t - u_{xx} = -f_1(w)\,u + f_2(w) \tag{3.18a}$$

$$v_t - v_{xx} = -f'_1(w)\,w_t u - f_1(w)\,v + f'_2(w)\,w_t \tag{3.18b}$$

$$w_t = -h_1(u)\,w + h_2(u) \tag{3.18c}$$

where $u_t = v$. We observe that we can define $v_0(x) = u''_0(x) - f_1(w_0(x))\,u_0(x) + f_2(w_0(x))$. If $(u_0, v_0, w_0) \in Y_1 = H^1(0, 1) \times L_2(0, 1) \times L_2(0, 1)$ we can define a semiflow given by the semigroup $\{S_0(t) \mid t \geqslant 0\}$ where

$$\begin{aligned} S_0(t)(u_0, v_0, w_0) &= (u(\cdot, t), v(\cdot, t), w(\cdot, t)) \\ &= (u(\cdot, t), u_t(\cdot, t), w(\cdot, t)) \qquad \text{for} \quad t \geqslant 0. \end{aligned}$$

Regularity results obtained from energy estimate argument ensure that $S_0(t)$: $Y_2 \to Y_2$. Following Hale and Raugel [24] we can extend $\mathscr{A}$ by defining

$$\mathscr{A}_0 = \{(\varphi, \psi, \theta) \mid \psi = \varphi'' - f_1(\theta)\,\varphi + f_2(\theta) \text{ where } (\varphi, \theta) \in \mathscr{A}\}. \tag{3.19}$$

We have

THEOREM 3.20. *If $(u_0, v_0, w_0) \in Y_2$ then the semiflow associated with the semigroup $\{S_0(t) \mid t \geqslant 0\}$ has global attractor $\mathscr{A}_0$ in Y_2,*

$$S_0(t)\,\mathscr{A}_0 = \mathscr{A}_0 \tag{3.21}$$

and

$$\lim_{t \to \infty} \delta_{Y_2}(S_0(t)\,B, \mathscr{A}_0) = 0 \textit{ for any bounded subset } B \subset Y_2. \tag{3.22}$$

4. THE HYPERBOLIC SINGULAR PERTURBATION

In this section we replace the leading equation (3.1a) of the parabolic Hodgkin–Huxley system with an equation containing a singular perturbation. We consider the system:

$$\varepsilon u_{tt} + (T + \varepsilon f_3(w))\,u_t = u_{xx} - f_2(w)\,u + f_2(w), \tag{4.1a}$$

$$w_t = -h_1(u)\,w + h_2(u), \qquad t > 0, \quad x \in (0, 1). \tag{4.1b}$$

Once again we impose homogeneous Neumann boundary conditions:

$$u_x(0, t) = u_x(1, t) = 0, \qquad t \geqslant 0. \tag{4.2}$$

Because we are now considering a second order hyperbolic equation we need another initial condition, i.e.:

$$u(x, 0) = u_0(x), \tag{4.3a}$$

$$u_t(x, 0) = v_0(x), \tag{4.3b}$$

$$w(x, 0) = w_0(x), \qquad x \in (0, 1). \tag{4.3c}$$

We notice that (4.1a) also involves a quasilinear perturbation of the first order time derivative. We therefore replace HH(iv) with

HH(iv′) $f_1(\cdot)$, $f_2(\cdot)$, $f_3(\cdot)$ are smooth nonnegative functions; they and their derivatives are polynomially bounded.

A minor adjustment of Theorem 4.7 of [17], and using Theorem 2 of [18], produces the following result which we state without proof.

THEOREM 4.4. *For each* $(u_0(\cdot), v_0(\cdot), w_0(\cdot)) \in Y_1 = H^1(0, 1) \times L_2(0, 1) \times L_2(0, 1)$ *with* $\|w_0\|_\infty < \infty$, *there exists a unique strong solution to* (4.1a–b), (4.2) *and* (4.3a–c) *on* $[0, 1] \times [0, \infty)$.

We set $v(\cdot, t) = u_t(\cdot, x)$ and define $\{S_\varepsilon(t) \mid t \geqslant 0\}$ on Y_1 by

$$S_\varepsilon(t)(u_0, v_0, w_0) = (u(\cdot, t), v(\cdot, t), w(\cdot, t)) \qquad \text{for} \quad t \geqslant 0. \tag{4.5}$$

We shall have occasion to speak of the semiflow σ_ε associated with $\{S_\varepsilon(t) \mid t \geqslant 0\}$, i.e.

$$\sigma_\varepsilon(t, u_0, v_0, w_0) = S_\varepsilon(t)(u_0, v_0, w_0). \tag{4.6}$$

It shall be useful to write our solution triple as the solution of an abstract semilinear Cauchy initial value problem. With $\varepsilon > 0$ we define G_ε: $Y_1 \to Y_1$ by the operator matrix

$$G_\varepsilon = \begin{pmatrix} 0 & I & 0 \\ -(1/\varepsilon)\, A & -(1/\varepsilon)\, I & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.7a}$$

with

$$D(G_\varepsilon) = D(A) \times D(A^{1/2}) \times L_2(0, 1). \tag{4.7b}$$

This operator, cf. Goldstein [19], is known to be the infinitesimal generator of a strongly continuous semigroup on Y_1, denoted by $\{T_\varepsilon(t) \mid t \geqslant 0\}$. We further remark that $\{T_\varepsilon(t) \mid t \geqslant 0\}$ can be viewed as a semigroup mapping Y_2 to itself. A nonlinear mapping is defined by writing

$$F_\varepsilon(u, v, w) = \begin{pmatrix} 0 \\ 1/\varepsilon(-f_1(w)\,u + f_2(w) - \varepsilon f_3(w)\,v) \\ -h_1(u)\,w + h_2(u) \end{pmatrix}. \tag{4.8}$$

Then if $U(t) = (u(\cdot, t),\ v(\cdot, t),\ w(\cdot, t))$, $U_0 = (u_0, v_0, w_0)$, solutions to (4.1a–b), (4.2), (4.3a–c) may be viewed as solutions to the abstract differential equation

$$dU/dt = G_\varepsilon U + F_\varepsilon(U) \tag{4.9a}$$

$$U(0) = U_0. \tag{4.9b}$$

We now turn our attention toward the question of attractors for the semiflow σ_ε associated with $\{S_\varepsilon(t) \mid t \geqslant 0\}$. In addition to the hypotheses HH(i)–(iii), (iv′), (v), we require that

HH(vi) $\quad w^* \geqslant w_0(x) \geqslant 0$ for $x \in [0, 1]$, where $w^* > 0$ is a fixed constant.

We note that this assumption guarantees that the constants C_w and C_f in (3.8) and (3.9) are independent of $w_0(x)$, and remark that without this assumption, the question of existence of global attractors for σ_ε is open.

We state two results which are established in [18].

THEOREM 4.10. *There is an $\varepsilon_0 > 0$ so that if $0 < \varepsilon \leqslant \varepsilon_0$ the semiflow σ_ε associated with $\{S_\varepsilon(t) \mid t \geqslant 0\}$ has an absorbing ball B_R in the space Y_1.*

THEOREM 4.11. *There exists an $\varepsilon_0 > 0$ so that if $0 < \varepsilon \leqslant \varepsilon_0$ the semiflow σ_ε has a global attractor $\mathscr{A}_\varepsilon$ in Y_1.*

We shall not reiterate the proofs of Theorems 4.10 and 4.11 which appear in [18, Theorems 2, 3]. However, we point out that in the course of establishing Theorem 4.10 we employ an energy estimate argument multiplying equation (4.1a) by an expression of the form $u_t + \rho u$ with $\rho > 0$ and integrating to produce constants C_1 and $K > 0$ so that

$$\begin{aligned} \|u(\cdot, t), u_t(\cdot, t)\|^2_{X_1} &= \|u(\cdot, t), v(\cdot, t)\|^2_{X_1} \\ &\leqslant \frac{2}{\varepsilon} \max\{\varepsilon_0, 1, \rho + \rho^2\}\ \|(u_0, u_1)\|^2_{X_1} e^{-2Kt} \\ &\quad + C_1/\varepsilon K, \qquad t \geqslant 0. \end{aligned} \tag{4.12}$$

We can also use the integral representation formula for solutions to (4.1b) (cf. (3.7), (3.8) and HH(vi)) to produce a $C_w > 0$ so that

$$\|w(\cdot, t)\| \leqslant C_w. \tag{4.13}$$

We can then assert that the ball B_R centered at the origin of Y_1 with radius

$$R = \sqrt{C_1/\varepsilon K + (C_w)^2} \tag{4.14}$$

is an absorbing ball for the semiflow σ_ε.

The existence result for the global attractor relies upon Theorem 2.4 where the decomposition $S_\varepsilon(t) = S_{\varepsilon_1}(t) + S_{\varepsilon_2}(t)$ is given as follows:

$$S_{\varepsilon_1}(t)(u_0, v_0, w_0) = (u_1(\cdot, t), v_1(\cdot, t), w_1(\cdot, t)) \tag{4.15}$$

where $\partial/\partial t(u_1(\cdot, t)) = v_1(\cdot, t))$ and $u_1(\cdot, t), w_1(\cdot, t)$ solves the linear system

$$\varepsilon u_{1tt} + (1 + \varepsilon f_3(w))\, u_{1t} - u_{1xx} + a u_1 = 0 \tag{4.16a}$$

$$w_{1t} = -h_1(u)\, w_1 \tag{4.16b}$$

with

$$u_1(x, 0) = u_0, \qquad u_{1t}(x, 0) = v_0 \tag{4.16c}$$

$$w_1(x, 0) = w_0, \tag{4.16d}$$

and

$$S_{\varepsilon_2}(t)(u_0, v_0, w_0) = (u_2(\cdot, t), v_2(\cdot, t), w_2(\cdot, t)), \tag{4.17}$$

where $\partial/\partial t(u_2(\cdot, t)) = v_2(\cdot, t)$ and $u_2(\cdot, t), w_2(\cdot, t)$ solves the inhomogeneous system

$$\varepsilon u_{2tt} + (1 + \varepsilon f_3(w))\, u_{2t} - u_{2xx} - a u_2 = -f_1(w)\, u + f_2(w) \tag{4.18a}$$

$$w_{2t} = -h_1(u)\, w_2 + h_2(u) \tag{4.18b}$$

with

$$u_2(x, 0) = 0, \qquad u_{2t}(x, 0) = 0 \tag{4.18c}$$

$$w_2(x, 0) = 0. \tag{4.18d}$$

Here, (cf. [18, Theorem 3]) $S_{\varepsilon_1}(t)$ is shown to satisfy the decay property (2.5a) and $S_{\varepsilon_2}(t)$ is shown to satisfy the eventual compactness property (2.5b).

We remark that this type of decomposition is somewhat common for the dynamics analysis of semilinear hyperbolic differential equations.

5. Regularity of the Attractors

Our computation of Hausdorff and fractal dimensions requires the solution semigroup differentiability and this differentiability necessitates our working in the smoother space Y_2. We shall show that the semigroup $\{S_\varepsilon(t) \mid t \geqslant 0\}$ acts on the space Y_2 and that the set $\mathscr{A}_\varepsilon$ lies in Y_2 and is an attractor for the semiflow associated with $\{S_\varepsilon(t) \mid t \geqslant 0\}$ in Y_2. Our first result follows.

Theorem 5.1. *Assume that there are positive constants r_1 and r_2 such that the initial data satisfies*

$$\|(u_0, v_0, w_0)\|_{Y_1} \leqslant r_1 \tag{5.2a}$$

and

$$\|(u_0, v_0, w_0)\|_{Y_2} \leqslant r_2. \tag{5.2b}$$

Then there exist positive constants ε_1, k_4, $C_1^*(r_2)$ and $C_2^*(r_1)$ such that for $0 < \varepsilon < \varepsilon_1$

$$\begin{aligned} \varepsilon \|u_{tt}\|^2 + \|(u, v, w)\|_{Y_2}^2 &= \varepsilon \|u_{tt}\|^2 + \|(u, u_t, w)\|_{Y_2}^2 \\ &\leqslant \frac{C_1^*(r_2)}{\varepsilon} \exp(-2k_4 t) + C_2^*(r_1). \end{aligned} \tag{5.3}$$

Proof. We begin by computing the L_2 inner product of the leading equation (4.1a) with $u_t + \rho u$, $\rho > 0$, to observe that

$$\begin{aligned} &\tfrac{1}{2} d/dt\{\varepsilon \|u_t\|^2 + \|u_x\|^2 + 2\varepsilon\rho\langle u_t, u\rangle + \rho \|u\|^2\} \\ &\quad + (1 - \varepsilon\rho) \|u_t\|^2 + \rho \|u_x\|^2 + \varepsilon \int_0^1 f_3(w)\, u_t^2\, dx \\ &\quad + \int_0^1 (f_1(w) + \varepsilon\rho f_3(w))\, u u_t\, dx \\ &\quad + \rho \int_0^1 f_1(w)\, u^2\, dx - \rho\langle f_2(w), u\rangle \\ &\quad - \langle f_2(w), u_t\rangle = 0. \end{aligned} \tag{5.4}$$

By (3.8) and HH(iv′) and (vi), we can adjust (3.9) to include the nonlinear term $f_3(w)$. Let

$$C_f \geqslant \sup_{t \geqslant 0} \{\|f_i(w(\cdot, t))\|_\infty \mid i = 1, 2, 3\}. \tag{5.5}$$

We define $\Gamma(t)$ by

$$\Gamma(t)=\varepsilon\ \|u_t\|^2+\|u_x\|^2+2\varepsilon\rho\langle u_t, u\rangle+\rho\ \|u\|^2 \tag{5.6}$$

and observe that, for a positive constant K_1,

$$\begin{aligned}
&(1-\varepsilon\rho)\ \|u_t\|^2+\rho\ \|u_x\|^2+\varepsilon\int_0^1 f_3(w)\, u_t^2\, dx\\
&\quad+\int_0^1 (f_1(w)+\varepsilon\rho f_3(w))\, uu_t\, dx+\rho\int_0^1 f_1(w)\, u^2\, dx\\
&\quad-\rho\langle f_2(w), u\rangle-\langle f_2(w), u_t\rangle-K_1\Gamma(t)\\
&\geqslant(1/2-\varepsilon\rho-\varepsilon K_1-\varepsilon C_f)\ \|u_t\|^2+(\rho-K_1)\ \|u_x\|^2\\
&\quad+(\rho a/2-\rho K_1-(C_f+\varepsilon\rho C_f+2\varepsilon\rho K_1)^2)\ \|u\|^2\\
&\quad-C_f^2\left(\frac{2}{a}+1\right)
\end{aligned} \tag{5.7}$$

where a is the constant of HH(i). We have two unspecified constants $K_1>0$ and $\rho>0$. We choose $\rho>2$ so that

$$\rho a/2-1-4(C_f+1)^2\geqslant 0. \tag{5.8}$$

Then choose and fix $K_1>0$ so that $K_1<\min\{\rho, 1/\rho\}\, v$ and then choose ε_0 so that $0<\varepsilon\leqslant\varepsilon_0$ and $\varepsilon_0>0$ is a constant such that $\varepsilon_0(\rho+(1/\rho)+C_f)\leqslant 1/2$, $\varepsilon_0\max\{\rho, 1\}\leqslant 1$, and $2\varepsilon_0\rho^2\leqslant 1$ (and hence $\rho-2\varepsilon\rho^2>1$). If we let

$$C_2=2(C_f)^2\left(\frac{2}{a}+1\right)$$

we can obtain

$$d/dt(\Gamma(t))+2K_1\Gamma(t)\leqslant C_2 \tag{5.9}$$

and hence

$$\Gamma(t)\leqslant\Gamma(0)\exp(-2K_1 t)+C_2/2K_1. \tag{5.10}$$

(The above analysis is similar to the proof of Theorem 2 in [18], and the reader is referred thereto for more details.)

Consequently we have

$$\frac{\varepsilon}{2}\ \|u_t\|^2+\|u_x\|^2+\|u\|\leqslant\Gamma(t)\leqslant\Gamma(0)\exp(-2K_1 t)+C_2/2K_1 \tag{5.11}$$

and

$$\frac{\varepsilon}{2}\|u_t\|^2+\|u_x\|^2+\|u\|^2+\|w\|^2\leqslant C_3(r_1)\exp(-2K_1t)+C_2/2K_1$$
$$\leqslant C_4(r_1) \tag{5.12}$$

whenever $\|(u_0, v_0, w_0)\|_{Y_1}\leqslant r_1$, where $C_4(r_1)=C_3(r_1)+C_2/2K_1$.

We now assume that $\|(u_0, v_0, w_0)\|_{Y_2}\leqslant r_2$ and recall that $\partial u/\partial t=v$. Therefore v satisfies the system

$$\varepsilon v_{tt}+v_t-v_{xx}=-\varepsilon f_3(w)\,v_t-\varepsilon f_3'(w)\,w_t v-f_1(w)\,v$$
$$-f_1'(w)\,w_t u+f_2'(w)\,w_t, \tag{5.13a}$$
$$w_t=-h_1(u)\,w+h_2(u), \qquad t>0, \quad x\in(0,1) \tag{5.13b}$$

with boundary conditions

$$v_x(0,t)=v_x(1,t)=0, \qquad t\geqslant 0 \tag{5.13c}$$

and initial conditions

$$v(x,0)=v_0(x), \qquad w(x,0)=w_0(x), \qquad x\in(0,1) \tag{5.13d}$$
$$v_t(x,0)=-\frac{1}{\varepsilon}\left[(1+\varepsilon f_3(w_0(x))\,v_0(x)-u_0''(x)\right.$$
$$\left.+f_1(w_0(x))\,u_0(x)-f_2(w_0(x))\right], \qquad x\in(0,1). \tag{5.13e}$$

Regularity results of Segal [41], also Webb [48], permit us to compute these derivatives and provide strong solutions to (5.13a–e) in Y_2. We compute the inner product of (5.13a) with $v_t+\rho v$ ($\rho>0$) to produce

$$\tfrac{1}{2}\,d/dt\,\{\varepsilon\,\|v_t\|^2+\|v_x\|^2+2\rho\varepsilon\langle v_t, v\rangle+\rho\,\|v\|^2\}$$
$$+\left\{(1-\varepsilon\rho)\,\|v_t\|^2+\rho\,\|v_x\|^2+\rho\int_0^1(\varepsilon f_3'(w)\,w_t+f_1(w))\,v^2\,dx\right.$$
$$+\varepsilon\int_0^1 f_3(w)\,v_t^2\,dx+\int_0^1(\varepsilon f_3'(w)\,w_t+f_1(w)+\varepsilon\rho f_3(w))\,vv_t\,dx$$
$$\left.+\int_0^1(f_1'(w)\,w_t u-f_2'(w)\,w_t)(v_t+\rho v)\,dx\right\}=0. \tag{5.14}$$

We set

$$\Gamma_1(t) = \varepsilon \|v_t\|^2 + \|v_x\|^2 + 2\varepsilon\rho\langle v_t, v\rangle + \rho \|v\|^2 \tag{5.15}$$

and let $M(t)$ denote the expression within the second pair of brackets on the left side of (5.14). If $K_2 > 0$ then we may observe that

$$\begin{aligned} M(t) - K_2 \Gamma_1(t) \geqslant{}& (1 - \varepsilon\rho - \varepsilon K_2) \|v_t\|^2 + (\rho - K_2) \|v_x\|^2 \\ &+ (\rho a - \rho K_2) \|v\|^2 + \varepsilon\rho \int_0^1 f_3'(w)\, w_t v^2 \, dx \\ &+ \varepsilon \int_0^1 f_3(w)\, v_t^2 + \int_0^1 (\varepsilon f'(w)\, w_t + f_1(w) \\ &+ \varepsilon\rho f_3(w) - 2\varepsilon\rho K_2)\, v v_t \, dx \\ &+ \int_0^1 (f_1'(w)\, w_t u - f_2'(w)\, w_t)(v_t + \rho v)\, dx. \end{aligned} \tag{5.16}$$

It is immediate from (5.13b), the hypothesis on $h_i(\cdot)$, $i = 1, 2$, and the bound on $\|w(\cdot, t)\|_\infty$ that there exists $C_5 > 0$ such that

$$\|w_t(\cdot, t)\| \leqslant C_5.$$

Thus combining (5.5) and (5.12) we obtain

$$\begin{aligned} M(t) - K_2 \Gamma_1(t) \geqslant{}& (1 - \varepsilon\rho - \varepsilon K_2) \|v_t\|^2 + (\rho - K_2) \|v_x\|^2 \\ &+ (\rho a - \rho K_2) \|v\|^2 - \varepsilon\rho C_f C_5 \|v\|^2 - \varepsilon C_f \|v_t\|^2 \\ &- (\varepsilon C_f C_5 + C_f + \varepsilon\rho C_f - 2\varepsilon\rho K_2) \|v\| \, \|v_t\| \\ &- C_f C_5 (1 + C_4(r_1))(\|v_t\| + \rho \|v\|). \end{aligned} \tag{5.17}$$

Because

$$\begin{aligned} &-(\varepsilon C_f C_5 + C_f + \varepsilon\rho C_f - 2\varepsilon\rho K_2) \|v\| \, \|v_t\| \\ &\qquad \geqslant -\tfrac{1}{4} \|v_t\|^2 - (C_f(\varepsilon C_5 + 1 + \varepsilon\rho) - 2\varepsilon\rho K_2) \|v\|^2, \end{aligned} \tag{5.18}$$

and

$$\begin{aligned} &-C_f C_5 (1 + C_4(r_1))(\|v_t\| + \rho \|v\|) \\ &\qquad \geqslant -\rho \frac{a}{2} \|v\|^2 - \frac{1}{4} \|v_t\|^2 - C_f^2 C_5^2 (1 + C_4(r_1)^2) \left(1 + \frac{\rho}{2a}\right), \end{aligned} \tag{5.19}$$

we can use (5.17) to observe that

$$M(t)-K_2\Gamma(t)\geqslant\left(\frac{1}{2}-\varepsilon\rho-\varepsilon K_2-\varepsilon C_f\right)\|v_t\|^2+(\rho-K_2)\|v_x\|^2$$

$$+\left[\frac{\rho a}{2}-\rho K_2-(C_f(\varepsilon C_5+1+\varepsilon\rho)-2\varepsilon\rho K_2)^2-\varepsilon\rho C_f C_5\right]\|v\|^2$$

$$-C_f^2C_5^2(1+C_4(r_1))^2\left(1+\frac{\rho}{2a}\right). \tag{5.20}$$

Moreover, ρ can be chosen sufficiently large and then K_2, and then ε_1, can be chosen sufficiently small so that

$$0<\varepsilon_1\leqslant\varepsilon_0, \qquad \rho-K_2\geqslant 0, \qquad 0<\rho K_2\leqslant 1, \qquad \rho>2, \tag{5.21}$$

$$\frac{\rho a}{2}-\rho K_2-(C_f(\varepsilon C_5+1+\varepsilon\rho)-2\varepsilon\rho K_2)^2-\varepsilon\rho C_f C_5\geqslant 0 \tag{5.22}$$

and

$$\tfrac{1}{2}-\varepsilon\rho-\varepsilon K_2-\varepsilon C_f\geqslant 0 \tag{5.23}$$

for $0<\varepsilon\leqslant\varepsilon_1$, where, in addition, ε_1 is chosen so that we have $\varepsilon_1\max\{2\rho^2, \rho, 1\}\leqslant 1$. From (5.14), (5.15) and (5.20) we thus have

$$\frac{d}{dt}\Gamma_1(t)+2K_2\Gamma_1(t)\leqslant C_6(r_1), \tag{5.24}$$

where $C_6(r_1)=C_f^2C_5^2(1+C_4(r_1))^2\,(1+(\rho/2a))$. (5.24) yields,

$$\Gamma_1(t)\leqslant\Gamma_1(0)\exp(-2K_2t)+\frac{C_6(r_1)}{2K_2},\ t\geqslant 0. \tag{5.25}$$

From (5.15) we have

$$\Gamma_1(0)=\varepsilon\,\|v_t(\cdot,0)\|^2+\|v_x(\cdot,0)\|^2+2\varepsilon\rho\langle v_t(\cdot,0), v_0(\cdot)\rangle+\rho\,\|v_0\|^2. \tag{5.26}$$

We know that,

$$\|v_x(\cdot,0)\|^2=\|v_0'\|^2, \qquad \|v(\cdot,0)\|^2=\|v_0\|^2. \tag{5.27}$$

Also, by Young's inequality,

$$\langle v_t(\cdot,0), v(\cdot,0)\rangle\leqslant\frac{1}{2\rho}\|v_t(\cdot,0)\|^2+\frac{\rho}{2}\|v(\cdot,0)\|^2$$

which implies that

$$2\varepsilon\rho\langle v_t(\cdot,0), v(\cdot,0)\rangle \leqslant \varepsilon\,\|v_t(\cdot,0)\|^2 + \varepsilon\rho^2\,\|v_0\|^2. \tag{5.28}$$

Because,

$$\begin{aligned}\varepsilon\,\|v_t(\cdot,0)\|^2 &= \varepsilon^{-1}\,\|1+f_3(w_0(\cdot))\,v_0(\cdot)-u_0''(\cdot)\\ &\quad +f_1(w_0(\cdot))\,u_0(\cdot)-f_2(w_0(\cdot))\|^2\\ &\leqslant \left(\frac{2}{\varepsilon}\right)[(1+\varepsilon C_f)^2\,\|v_0\|^2+\|u_0''\|^2+C_f^2\,\|u_0\|^2+C_f^2],\end{aligned}$$

(5.26)–(5.28) yield

$$\begin{aligned}\Gamma_1(0) &\leqslant \left(\left(\frac{4}{\varepsilon}\right)(1+\varepsilon C_f)^2+\varepsilon\rho^2+\rho\right)\|v_0\|^2+\|v_0'\|^2\\ &\quad +\left(\frac{4}{\varepsilon}\right)\|u_0''\|^2+\left(\frac{4}{\varepsilon}\right)C_f^2\,\|u_0\|^2+\left(\frac{4}{\varepsilon}\right)C_f^2.\end{aligned} \tag{5.29}$$

We define

$$C_7(r_2)=4[(1+\varepsilon C_f)^2+C_f^2+2]\,r_2^2+4C_f^2$$

and conclude via (5.25) and (5.29) that

$$\Gamma_1(t)\leqslant\left(\frac{1}{\varepsilon}\right)C_7(r_2)\exp(-2K_2t)+\frac{C_6(r_1)}{2K_2}, \qquad t\geqslant 0. \tag{5.30}$$

We refer to (5.15) to observe that, by our previous restrictions on ρ and ε,

$$\begin{aligned}\Gamma_1(t) &\geqslant \varepsilon\,\|v_t\|^2+\|v_x\|^2-2\varepsilon\rho\left(\frac{\|v_t^2\|}{4\rho}+\rho\,\|v\|^2\right)+\rho\,\|v\|^2\\ &\geqslant \frac{\varepsilon}{2}\,\|v_t\|^2+\|v_x\|^2+(\rho-2\varepsilon\rho^2)\,\|v\|^2\\ &\geqslant \frac{\varepsilon}{2}\,\|v_t^2\|+\|v_x\|^2+\|v\|^2,\end{aligned} \tag{5.31}$$

which together with (5.30) produces

$$\frac{\varepsilon}{2}\,\|u_{tt}\|^2+\|u_{tx}\|^2+\|u_t\|^2\leqslant\left(\frac{1}{\varepsilon}\right)C_7(r_2)\exp(-2K_2t)+\frac{C_6(r_1)}{2K_2}, \qquad t\geqslant 0. \tag{5.32}$$

From (4.1a) we observe that

$$u_{xx} = \varepsilon u_{tt} + (1 + \varepsilon f_3(w))\, u_t + f_1(w)\, u - f_2(w). \tag{5.33}$$

Returning to (5.12) we may couple (5.33) with (5.32) to produce the estimate

$$\|u_{xx}\|^2 \leqslant (1/\varepsilon)\, C_8(r_2) \exp(-2K_3 t) + C_9(r_1) \tag{5.34}$$

for positive constants $C_8(r_2)$, $C_9(r_1)$, and K_3.

We now differentiate our spatially dependent ordinary differential equation (4.1b) with respect to x to produce $C_{10}(r_1)$ such that

$$\|w_x(\cdot, t)\|^2 \leqslant 2e^{-2bt}\, \|w_0'\|^2 + C_{10}(r_1) \tag{5.35}$$

where we recall that b is the constant of HH(ii). The desired estimate will now follow by virtue of (5.12), (5.32), (5.34) and (5.35) and we conclude our argument.

We now want to establish that for sufficiently small ε the attractors $\mathscr{A}_\varepsilon$ belong to Y_2.

THEOREM 5.36. *There exists an $\varepsilon_2 > 0$ so that if ε is in $(0, \varepsilon_2]$ then $\mathscr{A}_\varepsilon \subset Y_2$ and $\mathscr{A}_\varepsilon$ is a global attractor for $\{S_\varepsilon(t) \mid t \geqslant 0\}$ in Y_2.*

Proof. We shall utilize the decomposition of $\{S_\varepsilon(t) \mid t \geqslant 0\}$,

$$S_\varepsilon(t) = S_{\varepsilon_1}(t) + S_{\varepsilon_2}(t) \tag{5.37}$$

where $S_{\varepsilon_1}(t)$ and $S_{\varepsilon_2}(t)$ are defined by (4.15) and (4.17) respectively. Our proof that $\mathscr{A}_\varepsilon \subset Y_2$ relies upon a result of Hale [22, Cor. 3.9.5]. Because we have shown the existence of absorbing sets and global attractors $\mathscr{A}_\varepsilon$ in Y_1 and $S_{\varepsilon_1}(t)$ satisfies the decay property (2.5a) in Y_1, it suffices to show that $S_{\varepsilon_2}(t)$ is conditionally completely continuous in Y_2, i.e., we need to show that if B is a bounded subset of Y_2 such that $\{S_{\varepsilon_2}(t) B \mid t \geqslant 0\}$ is bounded in Y_2, then $S_{\varepsilon_2}(t) B$ is precompact in Y_2.

In what follows we shall need to compute a priori estimates on some higher order derivatives of $u_2(\cdot, t)$ (as given by (4.17)). The regularity results which justify the formal computations follow from the classic paper of Segal [41]. Further application of the results also appear in Webb [48, 17]. To simplify matters we write $y(\cdot, t) = u_2(\cdot, t)$. It shall be our goal to produce an estimate of the form $\|(y(\cdot, t), y_t(\cdot, t), w_2(\cdot, t))\|_{Y_3} \leqslant \bar{C}_1(r)$ whenever $\|(u_0, v_0, w_0)\|_{Y_2} \leqslant r$. Here we recall that $Y_3 = H^3(0, 1) \times H^2(0, 1) \times H^2(0, 1)$. In the course of the proof of Theorem 3, [18], the authors established the existence of $\bar{C}_2(r)>0$ so that $\|(u_0, v_0, w_0)\|_{Y_1}\leqslant r$ implies that

$$\|(y(\cdot, t), y_t(\cdot, t), w_2(\cdot, t))\|_{Y_2}\leqslant \bar{C}_2(r). \tag{5.38}$$

If we multiply (4.18a) by y_{xxxxt} and integrate on (0, 1), we obtain

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{\varepsilon}{2}\|y_{xxt}\|^2+\frac{1}{2}\|y_{xxx}\|^2-\varepsilon\int_0^1 f_3(w)\,y_{xt}y_{xxx}\,dx\right.\\
&\quad-\varepsilon\int_0^1 f_3'(w)\,y_tw_xy_{xxx}\,dx-\int_0^1 f_1'(w)\,uw_xy_{xxx}\,dx\\
&\quad\left.-\int_0^1 f_1(w)\,u_xy_{xxx}\,dx+a\int_0^1 u_{1x}y_{xxx}\,dx-\int_0^1 f_2'(w)\,w_xy_{xxx}\,dx\right\}\\
&\quad+\|y_{xxt}\|^2+\varepsilon\int_0^1 f_3(u)\,y_{xtt}y_{xxx}\,dx+\varepsilon\int_0^1 f_3'(w)\,w_ty_{xt}y_{xxx}\,dx\\
&\quad+\varepsilon\int_0^1 f_3'(w)\,y_tw_{xt}\,y_{xxx}\,dx+\varepsilon\int_0^1 f_3'(w)\,y_{tt}w_xy_{xxx}\,dx\\
&\quad+\varepsilon\int_0^1 f_3''(w)\,w_ty_tw_xy_{xxx}\,dx+\int_0^1 f_1(w)\,u_{xt}y_{xxx}\,dx\\
&\quad+\int_0^1 f_1'(w)\,u_xw_ty_{xxx}\,dx\\
&\quad+\int_0^1 f_1'(w)\,uw_{xt}y_{xxx}\,dx+\int_0^1 f_1''(w)\,w_xw_ty_{xxx}\,dx+\int_0^1 f_1'(w)\,u_tw_xy_{xxx}\,dx\\
&\quad-a\int_0^1 u_{1xt}y_{xxx}\,dx-\int_0^1 f_2'(w)\,w_{xt}y_{xxx}\,dx-\int_0^1 f_2'(w)\,w_tw_xy_{xxx}\,dx=0.
\end{aligned} \tag{5.39}$$

Here we have used the fact that

$$\begin{aligned}
\langle f_3(w)\,y_t, y_{xxxt}\rangle &= -\int_0^1 y_{xxxt}(f_3(w)\,y_{tx}+y_tf_3'(w)\,w_x)\,dx\\
&= -\frac{d}{dt}\langle f_3(w)\,y_{xtt}, y_{xxx}\rangle+\langle f_3(w)\,y_{xt}, y_{xxx}\rangle\\
&\quad+\langle f_3'(w)\,w_ty_{xtt}, y_{xxx}\rangle-\frac{d}{dt}\langle f_3'(w)\,y_tw_x, y_{xxx}\rangle\\
&\quad+\langle f_3'(w)\,y_tw_{xt}, y_{xxx}\rangle+\langle \varepsilon f_3'(w)\,y_{tt}w_x, y_{xxx}\rangle\\
&\quad+\langle f_3''(w)\,w_ty_tw_x, y_{xxx}\rangle,
\end{aligned} \tag{5.40}$$

and used similar arguments for the terms $\langle f_1(w)\, u, y_{xxxxt}\rangle$ and $\langle f_2(w), y_{xxxxt}\rangle$.

We differentiate (4.18a) with respect to x to obtain

$$\varepsilon y_{xtt} + y_{xt} + \varepsilon f_3(w)\, y_{xt} + \varepsilon f_3'(w)\, w_x y_t - y_{xxx} + f_1(w)\, u_x + f_1'(w)\, w_x u - a u_{1x} - f_2'(w)\, w_x = 0. \tag{5.41}$$

For a positive constant η, we take the inner product of Eq. (5.41) with $-\eta y_{xxx}$ in $L_2(0, 1)$ to obtain

$$\begin{aligned}\varepsilon\eta \frac{d}{dt}\langle y_{xx}, y_{xxt}\rangle &- \varepsilon\eta\, \|y_{xxt}^2\| - \eta\langle y_{xt}, y_{xxx}\rangle \\ &- \eta\varepsilon\langle f_3(w)\, y_{xt}, y_{xxx}\rangle - \eta\varepsilon\langle f_3'(w)\, w_x y_t, y_{xxx}\rangle + \eta\, \|y_{xxx}\|^2 \\ &- \eta\langle f_1(w)\, u_x, y_{xxx}\rangle - \eta\langle f_1'(w)\, w_x u, y_{xxx}\rangle + \eta a\langle u_{1x}, y_{xxx}\rangle \\ &+ \eta\langle f_2'(w)\, w_x, y_{xxx}\rangle = 0.\end{aligned} \tag{5.42}$$

Summing up (5.39) and (5.42) we have

$$\begin{aligned}\frac{d}{dt}\Pi(t) &+ (1-\varepsilon\eta)\, \|y_{xxt}\|^2 + \eta\, \|y_{xxx}\|^2 \\ &\leqslant \|k(t; u, y, w, u_1)\|\, \|y_{xxx}\| + C_f\, \|y_{xxx}\|^2,\end{aligned} \tag{5.43}$$

where C_f is the constant of (5.5), $\Pi(t)$ is given by

$$\begin{aligned}\Pi(t) = \frac{\varepsilon}{2}\, \|y_{xxt}\|^2 &+ \frac{1}{2}\, \|y_{xxx}\|^2 + \varepsilon\eta\langle y_{xxt}, y_{xx}\rangle \\ &- \varepsilon\int_0^1 f_3(w)\, y_{xt} y_{xxx}\, dx - \varepsilon\int_0^1 f_3'(w)\, y_t w_x y_{xxx}\, dx - \int_0^1 f_1'(w)\, u w_x y_{xxx}\, dx \\ &- \int_0^1 f_1(w)\, u_x y_{xxx}\, dx + a\int_0^1 u_{1x} y_{xxx}\, dx - \int_0^1 f_2'(w)\, w_x y_{xxx}\, dx,\end{aligned} \tag{5.44}$$

and $k(t; u, y, w, u_1)$ is given by

$$\begin{aligned}k(t; u, y, w, u_1) = {}& f_3'(w)[\varepsilon w_t y_{xt} + \varepsilon y_t w_{xt} + \varepsilon y_{tt} w_x] \\ &+ f_3(w)[-y_{xt} - \varepsilon f_3(w)\, y_{xt} - \varepsilon f_3'(w)\, w_x y_t - f_1(w)\, u_x \\ &- f_1'(w)\, w_x u + a u_{1x} + f_2'(w)\, w_x] + \varepsilon f_3''(w)\, w_t y_t w_x + f_1(w)\, u_{xt} \\ &+ f_1'(w)[u_x w_t + u w_{xt} + u_t w_x]\end{aligned}$$

$$+f_1''(w)\,w_x w_t - a u_{1xt} - f_2'(w)\,w_{xt} - f_2''(w)\,w_t w_x$$
$$-\eta[\,y_{xt} + \varepsilon f_3(w)\,y_{xt} + \varepsilon f_3'(w)\,w_x y_t + f_1(w)\,u_x$$
$$+f_1'(w)\,w_x u - a u_{1x} - f_2'(w)\,w_x]. \tag{5.45}$$

Here we have used the equivalent form of εy_{xtt} from Eq. (5.41).

From (5.12), (5.38) and the decay estimate on $\|(u_1, u_{1t}, w_1)\|$, we produce the bounds

$$\sup_{t\geqslant 0}\{\|u\|^2, \|u_x\|^2, \|u_t\|^2, \|w_2\|^2, \|w_{2x}\|^2, \|y\|^2, \|y_x\|^2,$$
$$\|y_{xx}\|^2, \|y_t\|^2, \|y_{xt}\|^2, \|u_1\|^2, \|u_{1x}\|^2\} \leqslant \bar{C}_3(\varepsilon, r) \tag{5.46}$$

for a positive constant $\bar{C}_3(\varepsilon, r)$. From the equation (4.18b) we obtain a bound on $\|w_{2t}\|$,

$$\|w_{2t}\| \leqslant \bar{C}_4(r). \tag{5.47}$$

By differentiating (4.18b) with respect to x, we get

$$\|w_{2xt}\| \leqslant \bar{C}_5(r). \tag{5.48}$$

If we return to (4.18a) we may use these bounds to provide a constant $\bar{C}_6(r)$ so that

$$\|\varepsilon y_{tt}\| \leqslant \bar{C}_6(r). \tag{5.49}$$

Moreover, the uniform estimate for w produces a $\bar{C}_7 > 0$ so that

$$\sup_{t\geqslant 0}\{\|f_i(w)\|_\infty, \|f_i'(w)\|_\infty, \|f_i''(w)\|_\infty \mid i = 1, 2, 3\} < \bar{C}_7, \tag{5.50}$$

and returning to (5.32) we are assured of a $\bar{C}_8(\varepsilon, r)$ so that

$$\|u_{xt}\| \leqslant \bar{C}_8(\varepsilon, r). \tag{5.51}$$

Because $\|u_{1xt}\| \leqslant \|u_{xt}\| + \|y_{xt}\|$, (5.46) and (5.51) imply that

$$\|u_{1xt}\| \leqslant \bar{C}_9(\varepsilon, r) \tag{5.52}$$

for some positive constant $\bar{C}_9(\varepsilon, r)$ which depends on ε and $r \geqslant \|(u_0, v_0, w_0)\|_{Y_2}$. If we combine estimates (5.46)–(5.52) we can construct a constant $\bar{C}_{10}(\varepsilon, r)$ so that

$$\sup_{t\geqslant 0}\|k(t; u, y, w, u_1)\| \leqslant \bar{C}_{10}(\varepsilon, r) \tag{5.53}$$

whenever $\|(u_0, v_0, w_0)\|_{Y_2} \leqslant r$.

Let us now choose η sufficiently large so that $\eta - C_f \geqslant \eta/2$ and then choose ε so that $0 < \varepsilon \leqslant \varepsilon_2$ where $\varepsilon_2 = \min\{\varepsilon_1, \eta/2\}$. By virtue of (5.43) we have

$$\frac{d}{dt}\Pi(t) + \frac{1}{2}\|y_{xxt}\|^2 + \frac{\eta}{2}\|y_{xxx}\|^2 \leqslant \bar{C}_{10}\|y_{xxx}\| \leqslant \frac{\eta}{4}\|y_{xxx}\|^2 + \frac{\bar{C}_{10}(\varepsilon, r)^2}{\eta}. \tag{5.54}$$

For $\delta > 0$, $\delta < \min\{1/\varepsilon_2, \eta/2\}$,

$$\begin{aligned}
&\tfrac{1}{2}[\|y_{xxt}\|^2 + \eta\|y_{xxx}\|^2] - \delta\Pi(t) \\
&\quad = \tfrac{1}{2}(1-\varepsilon\delta)\|y_{xxt}\|^2 + \tfrac{1}{2}(\eta-\delta)\|y_{xxx}\|^2 - \delta\varepsilon\eta\langle y_{xxt}, y_{xx}\rangle \\
&\qquad + \delta\varepsilon\int_0^1 f_3(w)\, y_{xt} y_{xxx}\, dx + \delta\varepsilon\int_0^1 f_3'(w)\, y_t w_x y_{xxx}\, dx \\
&\qquad + \delta\int_0^1 f_1'(w)\, u w_x y_{xxx}\, dx + \delta\int_0^1 f_1(w)\, u_x y_{xxx}\, dx \\
&\qquad - \delta\int_0^1 u_{1x} y_{xxx}\, dx + \delta\int_0^1 f_2'(w)\, w_x y_{xxx}\, dx \\
&\quad \geqslant -\delta\|L(t; u, y, w, u_1)\|\,\|y_{xxx}\|,
\end{aligned} \tag{5.55}$$

where

$$\begin{aligned}
L(t; u, y, w, u_1) &= \varepsilon f_3(w)\, y_{xt} + \varepsilon f_3'(w)\, y_t w_x + f_1'(w)\, u w_x \\
&\quad + f_1(w)\, u_x + u_{1x} + f_2'(w)\, w_x + \varepsilon\eta y_{xt}.
\end{aligned}$$

Similar to the above, there is a constant $\bar{C}_{11}(\varepsilon, r) > 0$ such that

$$\|L(t; u, y, w, u_1)\| \leqslant \bar{C}_{11}(\varepsilon, r) \tag{5.56}$$

for any $\|(u_0, v_0, w_0)\|_{Y_2} \leqslant r$. From (5.54)–(5.56) it follows that

$$\begin{aligned}
\frac{d}{dt}\Pi(t) + \delta\Pi(t) + \frac{\eta}{4}\|y_{xxx}\|^2 &\leqslant \delta\|L\|\,\|y_{xxx}\|^2 + \frac{\bar{C}_{10}(\varepsilon, r)^2}{\eta} \\
&\leqslant \frac{\eta}{4}\|y_{xxx}\|^2 + \frac{\delta^2\bar{C}_{11}(\varepsilon, r)^2}{\eta} + \frac{\bar{C}_{10}(\varepsilon, r)^2}{\eta},
\end{aligned} \tag{5.57}$$

or

$$\frac{d}{dt}\Pi(t) + \delta\Pi(t) \leqslant \frac{1}{\eta}[\bar{C}_{10}(\varepsilon, r)^2 + \delta^2\bar{C}_{11}(\varepsilon, r)^2]. \tag{5.58}$$

Because $\Pi(0) = 0$, we integrate (5.58) to observe that

$$\Pi(t) \leqslant \frac{1}{\delta\eta}[\bar{C}_{10}(\varepsilon, r)^2 + \delta^2\bar{C}_{11}(\varepsilon, r)^2], \qquad t \geqslant 0. \tag{5.59}$$

Therefore, from (5.44) we have

$$\frac{\varepsilon}{2}\|y_{xxt}\|^2+\frac{1}{2}\|y_{xxx}\|^2\leqslant\bigg|\varepsilon\int_0^1 f_3(w)\,y_{xt}y_{xxx}\,dx+\varepsilon\int_0^1 f_3'(w)\,y_tw_xy_{xxx}\,dx$$
$$+\int_0^1 f_1'(w)\,uw_xy_{xxx}\,dx+\int_0^1 f_1(w)\,u_xy_{xxx}\,dx$$
$$-a\int_0^1 u_{1x}y_{xxx}\,dx+\int_0^1 f_2'(w)\,w_xy_{xxx}\,dx$$
$$-\varepsilon\eta\int_0^1 y_{xxt}y_{xx}\,dx+\frac{1}{\delta\eta}\,[\bar{C}_{10}(\varepsilon,r)^2+\delta^2\bar{C}_{11}(\varepsilon,r)^2]\bigg|. \tag{5.60}$$

By using Young's inequality to handle the integrals in (5.60), we find that

$$\|y_{xxt}\|^2+\|y_{xxx}\|^2\leqslant C^*(\varepsilon,r), \tag{5.61}$$

and hence, there is a positive constant $C^{**}(\varepsilon,r)$ so that

$$\|(y(t),y_t(t))\|^2_{X_3}\leqslant C^{**}(\varepsilon,r). \tag{5.62}$$

Finally, to obtain a bound on w_{2xx}, we differentiate Eq. (4.18b) twice with respect to x to obtain

$$w_{2xxt}=-h_1(u)\,w_{2xx}-h_1'(u)\,u_xw_{2x}-h_1'(u)\,u_{xx}w_2$$
$$-h_1''(u)\,u_x^2w_2+h_2'(u)\,u_{xx}+h_2''(u)\,u_x^2. \tag{5.63}$$

Forming the inner product of (5.63) with w_{2xx}, we obtain

$$\frac{1}{2}\frac{d}{dt}\|w_{2xx}\|^2+b\,\|w_{2xx}\|^2\leqslant\frac{1}{2}\frac{d}{dt}\|w_{2xx}\|^2+\int_0^1 h_1(u)\,w_{2xx}^2\,dx$$
$$\leqslant|\langle h_1'(u)\,u_xw_{2x}+h_1'(u)\,u_{xx}w_2$$
$$+h_1''(u)\,u_x^2w_2+h_2'(u)\,u_{xx}+h_2''(u)\,u_x^2,\,w_{2xx}\rangle|$$
$$\leqslant\frac{b}{2}\|w_{2xx}\|^2+\frac{1}{2b}\bar{C}_{12}^2(\varepsilon,r), \tag{5.64}$$

where $\bar{C}_{12}(\varepsilon,r)$ is a constant such that if $\|(u_0,v_0,w_0)\|_{Y_2}\leqslant r$ then

$$\|h_1'(u)\|\,\|u_x\|\,\|w_{2x}\|+\|h_1'(u)\|\,\|u_{xx}\|\,\|w_2\|+\|h_1''(u)\|\,\|u_x\|^2\,\|w_2\|$$
$$+\|h_2'(u)\|\,\|u_{xx}\|+\|h_2''(u)\|\,\|u_x\|^2\leqslant\bar{C}_{12}(\varepsilon,r). \tag{5.65}$$

Thus, there is a constant $C^{***}(\varepsilon, r) > 0$ such that

$$(\|w(\cdot, t)\|^{(2)})^2 = \|w_2\|^2 + \|w_{2x}\|^2 + \|w_{2xx}\|^2 \leqslant C^{***}(\varepsilon, r). \qquad (5.66)$$

If B_r is a ball of radius r in Y_2 centered at the origin we may combine (5.62) and (5.66) to observe that

$$\bigcup_{t \geqslant 0} S_{\varepsilon_2}(t)\, B_r \subseteq B_{r^*} \qquad (5.67)$$

where B_{r^*} is a bounded ball centered at the origin in Y_3 of radius $r^* = (C^{**}(\varepsilon, r) + C^{***}(\varepsilon, r))^{1/2}$. The Sobolev Embedding Theorem guarantees that Y_3 is compactly embedded in Y_2 and consequently the hypotheses of the aforementioned Corollary 3.9.5 of Hale, [22], are satisfied and we thereby conclude that $\mathscr{A}_\varepsilon \subset Y_2$ is a global attractor for the semiflow associated with $\{S_\varepsilon(t) \mid t \geqslant 0\}$.

In the manner of Corollaries 2.6 and 2.7 of Hale and Raugel [24] we may apply the preceding results, Theorems 4.10, 5.1, 5.36 and the invariance property of attractors to obtain the following two corollaries which we simply state.

COROLLARY 5.68. *Let* $0 < \varepsilon \leqslant \varepsilon_1$. *Then system* (4.1a–b), (4.2), (4.3a–c) *is bounded dissipative in* Y_2 *uniformly in* ε. *By this we mean that there exists a bounded set* $B \subset Y_2$ *such that for any bounded* $U \subset Y_2$ *there exists a positive number* $t_*^\varepsilon = t_*(U, B, \varepsilon)$ *such that* $t \geqslant t_*^\varepsilon$ *implies* $S_\varepsilon(t) U \subseteq B$.

COROLLARY 5.69. *Let* $0 < \varepsilon \leqslant \varepsilon_2$. *Then there exists a positive constant* k_1 *so that if* $(\varphi, \psi, \theta) \in \mathscr{A}_\varepsilon$ *then* $\|(\varphi, \psi, \theta)\|_{Y_2} \leqslant k_1$. *Moreover* k_2 *can be chosen so that if* $0 < \varepsilon \leqslant \varepsilon_2$, *then for any solution triple* $U_\varepsilon(t) = (u_\varepsilon(\cdot, t), \partial u_\varepsilon(\cdot, t)/\partial t, w(\cdot, t))$ *with* $U_\varepsilon(\mathbb{R}^+) \subset \mathscr{A}_\varepsilon$ *we have*

$$\sqrt{\varepsilon}\, \|\partial^2 u_\varepsilon(\cdot, t)/\partial t^2\| \leqslant k_2. \qquad (5.70)$$

We return to the comments immediately preceding the statement of Theorem 3.17 regarding the regularity of the attractor $\mathscr{A}$ of the unperturbed Hodgkin–Huxley system. Parabolic theory allows us to produce uniform L_2 estimates for u, u_t, u_x, u_{xx}, u_{tt}, u_{xt}, w, w_x, w_t for solution pairs $(u(\cdot), w(\cdot))$ for initial data in X_2. Moreover we can apply these estimates to obtain uniform higher order estimates on $u_2(\cdot, t)$ satisfying

$$u_{2t} = u_{2xx} + au_1 - f_1(w)\, u + f_2(w) \qquad (5.71a)$$

$$u_2(x, 0) = 0 \qquad (5.71b)$$

where $u_2+u_1=u$. For any $\varepsilon>0$ we can rewrite (5.71a) as

$$\varepsilon u_{2tt}-u_{2t}+u_{2xx}+au_1=f_1(w)\,u-f_2(w)+\varepsilon u_{2tt}$$

and use the hyperbolic arguments above taking advantage of uniform estimates for the derivative u_{tt}. We leave this straightforward adaption to readers.

6. Differentiability of the Solution Semigroup

Corollary 5.68 establishes the existence of a uniform absorbing set B for the semiflows $\sigma_\varepsilon(t; u_0, v_0, w_0)$ in Y_2 for sufficiently small $\varepsilon>0$. Because we are interested in the longterm behavior of solutions it is therefore sufficient to confine our attention to the flow in the absorbing set B.

In light of the foregoing remarks we truncate our nonlinearities beyond the absorbing set B. To be more precise we let B_R be a ball of sufficiently large radius R in $Y_2=H^2(0,1)\times H^1(0,1)\times H^1(0,1)$ so that $B\subseteq B_R$. Then if $(u, v, w)\in B\subseteq B_R$,

$$\sup\{\|u\|_\infty, \|w\|_\infty\}\leqslant R. \tag{6.1}$$

We mollify the characteristic function of the interval $[-R, R]$ by letting $\rho(\cdot)$ be a function so that

$$\begin{array}{lll} \text{(i)} & \rho\in C^\infty(\mathbb{R}) & \\ \text{(ii)} & \rho(y)=1 & \text{for} \quad -R\leqslant y\leqslant R \\ \text{(iii)} & \rho(y)=0 & \text{for} \quad |y|\geqslant R+1 \\ \text{(iv)} & \rho'(y)\geqslant 0 & \text{for} \quad -(R+1)\leqslant y\leqslant -R \\ \text{(v)} & \rho'(y)\leqslant 0 & \text{for} \quad R\leqslant y\leqslant R+1, \end{array} \tag{6.2}$$

and defining

$$\tilde f_i(u)=\rho(\|u\|_\infty)\,f_i(u) \qquad i=1, 2, 3 \tag{6.3a}$$

$$\tilde h_i(u)=\rho(\|u\|_\infty)\,h_i(u) \qquad i=1, 2 \tag{6.3b}$$

We now consider the modified system whose strong solution is guaranteed by Theorem (4.4):

$$\varepsilon u_{tt}+(1+\varepsilon\tilde f_3(w))\,u_t-u_{xx}=-\tilde f_1(w)\,u+\tilde f_2(w), \tag{6.4a}$$

$$w_t=-\tilde h_1(u)\,w+\tilde h_2(u), \qquad t>0, \quad x\in(0,1) \tag{6.4b}$$

with boundary conditions

$$u_x(0, t) = u_x(1, t) = 0, \qquad t \geqslant 0 \tag{6.4c}$$

and initial conditions

$$\begin{aligned} u(x, 0) &= u_0(x) \\ u_t(x, 0) &= v_0(x) \\ w(y, 0) &= w_0(x), \qquad x \in (0, 1) \end{aligned} \tag{6.4d}$$

with

$$(u_0, v_0, w_0) \in Y_2. \tag{6.4e}$$

The following proposition should be self evident.

PROPOSITION 6.5. *If $(u_0, v_0, w_0) \in Y_2$ and the strong solution (u, v, w) of (4.1a–b), (4.2), (4.3a–c) $\subset B \subset Y_2$, then it agrees with the strong solution of (6.4a–e).*

Moreover, all the results established previously also hold for the system with truncated kinetics. Henceforth we restrict our attention to system (6.4a–e) and we shall use the notation $\tilde{\sigma}_\varepsilon$ and $\{\tilde{S}_\varepsilon(t) \mid t \geqslant 0\}$ to denote the semiflow and its associated semigroup. Finally we let $\tilde{F}_\varepsilon$ be the nonlinear mapping defined via (4.8) using the truncated nonlinearities.

It shall be our intention to establish the Fréchet differentiability of the semigroup $\{\tilde{S}_\varepsilon(t) \mid t \geqslant 0\}$. We shall introduce two lemmas which we for reasons of brevity state without proof. The results are not unexpected and the proofs, albeit lengthy and complicated, follow by routine arguments.

LEMMA 6.6. *The mapping $\tilde{F}_\varepsilon\colon Y_2 \to Y_2$ is Fréchet differentiable at $U = (u, v, w) \in Y_2$ with Fréchet derivative $\tilde{F}'_\varepsilon(U) \in \mathscr{L}(Y_2)$ given by*

$$\tilde{F}'_\varepsilon(U)\, r = \tilde{F}_\varepsilon(u, v, w) \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ (1/\varepsilon)[-\tilde{f}_1(w)\, r_1 - \tilde{f}'_1(w)\, u r_3 + \tilde{f}'_2(w)\, r_3 \\ -\varepsilon \tilde{f}_3(w)\, r_2 - \varepsilon \tilde{f}'_3(w)\, v r_2] \\ -\tilde{h}_1(u)\, r_3 - \tilde{h}'_1(u)\, w r_1 + \tilde{h}'_2(u)\, r_1 \end{pmatrix} \tag{6.7}$$

for $r = (r_1, r_2, r_3) \in Y_2$.

LEMMA 6.8. *If Z is a bounded subset of Y_1 and $T>0$, there exists $l_1 = l_1(Z, T)$ so that for $U_1 = (u_{1_0}, v_{1_0}, w_{1_0})$, $U_2 = (u_{2_0}, v_{2_0}, w_{2_0})$ belonging to Z,*

$$\|\tilde{S}_\varepsilon(t)\, U_1 - \tilde{S}_\varepsilon(t)\, U_2\|_{Y_1} \leqslant (l_1/\varepsilon)\, \|U_1 - U_2\|_{Y_1}, \qquad t \in [0, T]. \tag{6.9}$$

Moreover, if Z is a bounded subset of Y_2, there exists $l_2 = l_2(Z, T)$ so that for $U_1, U_2 \in Z$,

$$\|\tilde{S}_\varepsilon(t)\, U_1 - \tilde{S}_\varepsilon(t)\, U_2\|_{Y_2} \leqslant (l_2/\varepsilon)\, \|U_1 - U_2\|_{Y_2}, \qquad t \in [0, T]. \tag{6.10}$$

In order to show the spatial differentiability of the solution semigroup $\tilde{S}_\varepsilon(t)$ and the variational equation satisfied by its Fréchet derivative, we prefer to prove a rather general theorem below, which will directly lead to the concerned differentiability result.

It is now convenient to work in a general Hilbert space setting. We let H be a real Hilbert space with norm $|\ |$. We consider the semilinear initial value problem

$$dy/dt = Gy + Fy, \qquad t > 0, \tag{6.11a}$$

$$y(0) = y_0, \tag{6.11b}$$

where G is the infinitesimal generator of a strongly continuous semigroup of nonexpansive operators $\{T(t) \mid t \geqslant 0\}$ and $F\colon H \to H$ is a nonlinear and locally Lipschitz continuous mapping. It is well known, cf. Pazy [39] or Goldstein [19], that strong solutions to (6.11a–b) exist and on their intervals of existence satisfy the integral equation,

$$y(t) = T(t)\, y_0 + \int_0^t T(t-s)\, F(y(s))\, ds. \tag{6.12}$$

We shall assume that strong solutions to (6.11a–b) are globally defined. Consequently, we have a globally defined solution semigroup $\{S(t) \mid t \geqslant 0\}$ where

$$S(t)\, y_0 = y(t) \tag{6.13}$$

and $y(\cdot)$ is the strong solution to (6.11a–b). We have the following result which is related to results appearing in Goldstein, Oharu and Takahashi [20].

THEOREM 6.14. *Assume that globally defined strong solutions to* (6.11a–b) *exist for any $y_0 \in H$ and have the abstract variation-of-parameters representation* (6.12). *We further assume that the following hold*:

(i) *The mapping* $F: H \to H$ *is Fréchet differentiable and its Fréchet derivative at* $y \in H$ *is denoted by* $F'(y)$,

(ii) *For each* $y \in H$ *the mapping* $F'(S(t)\,y): [0, \infty) \to \mathscr{L}(H)$ *is strongly measurable.*

(iii) *The mapping* $F'(S(t)\,y): [0, \infty) \times H \to \mathscr{L}(H)$ *is locally bounded.*

(iv) *For any bounded set* $B \subset H$ *and* $T > 0$ *there exists a* $K = K(B, T)$ *so that for* $y_1, y_2 \in B$,

$$|S(t)\,y_1 - S(t)\,y_2| \leqslant K\,|y_1 - y_2| \qquad \textit{for} \quad t \in [0, T]. \tag{6.14}$$

Then the solution semigroup $S(t): y_0 \to S(t)\,y_0$ *is Fréchet differentiable and its Fréchet derivative* $L(t, y_0)$ *satisfies the following nonautonomous linear evolution equation in* $\mathscr{L}(H)$:

$$\frac{d}{dt} L(t, y_0) = GL(t, y_0) + F'(S(t)\,y_0)\,L(t, y_0), \qquad t > 0 \tag{6.15}$$

$$L(0, y_0) = I_H. \tag{6.16}$$

Proof. We define the difference

$$\Delta(t, y, \zeta) = S(t)(y + \zeta) - S(t)\,y \tag{6.17}$$

for $t \geqslant 0$, $y, \zeta \in H$. Using assumption (iv), given any $y \in H$ and open bounded neighborhood $B(y)$ of y there is a constant $l = l(T, B(y)) > 0$ such that for $t \in [0, T]$ and $\zeta \in B(y)$ we have

$$|S(t)(y + \zeta) - S(t)\,y| \leqslant l\,|\zeta|. \tag{6.18}$$

It can be seen that $\Delta(t, y, \zeta)$ satisfies the following integral equation

$$\Delta(t, y, \zeta) = T(t)\,\zeta + \int_0^t T(t - \sigma)\,F'(S(\sigma)\,y)\,\Delta(\sigma, y, \zeta)\,d\sigma + R(t), \qquad t \geqslant 0, \tag{6.19}$$

where the remainder $R(t)$ is given by

$$R(t) = \int_0^t T(t - \sigma)\,|\Delta(\sigma, y, \zeta)|\,r(S(\sigma)\,y, \Delta(\sigma, y, \zeta))\,d\sigma, \qquad t \geqslant 0, \tag{6.20}$$

in which $r(z, \eta)$ is defined by the following differential relation,

$$F(z + \eta) - F(z) = F'(z)\,\eta + |\eta|\,r(z, \eta), \tag{6.21}$$

with the property that

$$\lim_{|\eta| \to 0} r(z, \eta) = 0, \qquad \text{for} \quad z \in H. \tag{6.22}$$

On the other hand, we know that we can define $g(t) = g(t, y, \zeta)$ to be the solution of the following equation,

$$g(t) = T(t)\,\zeta + \int_0^t T(t-\sigma)\, F'(S(\sigma)\, y)\, g(\sigma)\, d\sigma, \qquad t \geqslant 0. \tag{6.23}$$

We set

$$L(t, y)\,\zeta = g(t, y, \zeta) \tag{6.24}$$

and denote

$$h(t) = \Delta(t, y, \zeta) - L(t, y)\,\zeta = \Delta(t, y, \zeta) - g(t, y, \zeta), \qquad t \geqslant 0. \tag{6.25}$$

Then $h(\cdot)$ satisfies the following equation,

$$h(t) = \int_0^t T(t-\sigma)\, F'(S(\sigma)\, y)\, h(\sigma)\, d\sigma + R(t), \qquad t \geqslant 0. \tag{6.26}$$

Letting $\|\ \| = \|\ \|_{\mathscr{L}(H)}$, we have the estimate

$$|h(t)| \leqslant \int_0^t \|T(t-\sigma)\|\, \|F'(S(\sigma)\, y)\|\, |h(\sigma)|\, d\sigma + |R(t)|, \tag{6.27}$$

where, by the assumption (ii), the Volterra integral makes sense.

Let

$$k = k(T) = \sup_{0 \leqslant t \leqslant T} \|T(t)\|, \tag{6.28}$$

and

$$\alpha = \alpha(T, B(y)) = \sup_{\substack{0 \leqslant t \leqslant T \\ z \in B(y)}} \|F'(S(t)\, z)\|. \tag{6.29}$$

By the assumption (iii), we have $\alpha < \infty$, for any finite $T \geqslant 0$ and any bounded neighborhood $B(y)$ in H. Now, using (6.28) and (6.18), we get

$$\begin{aligned} |R(t)| &\leqslant \int_0^t \|T(t-\sigma)\|\, |\Delta(\sigma, y, \zeta)|\, |r(S(\sigma)\, y, \Delta(\sigma, y, \zeta))|\, d\sigma \\ &\leqslant kl\, |\zeta| \int_0^t |r(S(\sigma)\, y, \Delta(\sigma, y, \zeta))|\, d\sigma, \qquad \text{for} \quad t \in [0, T]. \end{aligned} \tag{6.30}$$

Since the Lipschitz property (6.18) implies that

$$\lim_{|\zeta|\to 0} |\Delta(\sigma, z, \zeta)| = 0, \tag{6.31}$$

in which the convergence is uniform in $z \in B(y)$ and in $\sigma \in [0, T]$, then according to (6.21) and (6.22), we have

$$\lim_{|\zeta|\to 0} |r(S(\sigma)\, y, \Delta(\sigma, y, \zeta))| = 0, \tag{6.32}$$

for any given $\sigma \in [0, T]$ and any given $y \in H$.

Moreover, we have by definition (6.21), that as $|\zeta| \to 0$,

$$|r(S(\sigma)\, y, \delta(\sigma, y, \zeta))| \leqslant \gamma(y, T) + \delta(y, T), \qquad \forall \sigma \in [0, T], \tag{6.33}$$

where $\gamma(y, T) \geqslant 0$ is a constant from the following relation

$$|F(S(\sigma)\, y + \psi) - F(S(\sigma)\, y)| \leqslant \gamma(y, T)|\psi|, \qquad \sigma \in [0, T], \tag{6.34}$$

by the locally Lipschitz continuous property of F, and $\delta(y, T) \geqslant 0$ is a constant determined by

$$\|F'(S(\sigma)\, y)\| \leqslant \delta(y, T), \qquad \sigma \in [0, T] \tag{6.35}$$

which is implied by assumption (iii). Now the assertions of (6.32) and (6.33) allow us to apply the Lebesgue Dominated Convergence Theorem in (6.30) to obtain

$$\begin{aligned} \sup_{0 \leqslant t \leqslant T} |R(t)| &\leqslant kl\, |\zeta| \int_0^T |r(S(\sigma)\, y, \Delta(\sigma, y, \zeta))|\, d\sigma \\ &= o(|\zeta|), \qquad \text{as} \quad |\zeta| \to 0. \end{aligned} \tag{6.36}$$

Finally, substituting (6.36) into (6.27), we get

$$|h(t)| \leqslant k\alpha \int_0^t |h(\sigma)|\, d\sigma + \sup_{0 \leqslant t \leqslant T} |R(t)|, \qquad t \in [0, t]. \tag{6.37}$$

By the Gronwall inequality, it follows that

$$|h(t)| \leqslant e^{k\alpha T} o(|\zeta|), \qquad \text{for} \quad t \in [0, T], \qquad \text{as} \quad |\zeta| \to 0. \tag{6.38}$$

This means that

$$|S(t)(y + \zeta) - S(t)\, y - L(t, y)\, \zeta| = o(|\zeta|), \qquad \text{as} \quad |\zeta| \to 0, \tag{6.39}$$

for $0 \leqslant t \leqslant T$. Therefore we have established the spatial Fréchet differentiability of $S(t)$ and the validity of (6.16a).

We remark that Theorem 6.14 is essentially an abstract semilinear variant of classical differentiability results of ordinary differential equations, cf. Coddington and Levinson [5] or Hartman [25].

We now return to the case at hand. For the remainder of the paper we shall be working in the specific function spaces defined in the second section. It should be clear that strong solutions to (6.4a–e) can be viewed as solutions to the abstract semilinear Cauchy initial value problem

$$dU/dt = G_\varepsilon U + \tilde{F}_\varepsilon(U), \qquad t > 0, \tag{6.40a}$$

$$U(0) = U_0 = (u_0, v_0, w_0), \tag{6.40b}$$

where G_ε is the linear operator defined by (4.7a–b) and $\tilde{F}_\varepsilon$ is defined via (4.8) using the truncated nonlinearities (6.3a–b). If $\{T_\varepsilon(t) \mid t \geqslant 0\}$ is the semigroup generated by G_ε then the solution semigroup $\{\tilde{S}_\varepsilon(t) \mid t \geqslant 0\}$ has the abstract variation of parameters representation,

$$\tilde{S}_\varepsilon(t)\, U_0 = T_\varepsilon(t)\, U_0 + \int_0^t T_\varepsilon(t-s)\, \tilde{F}_\varepsilon(U(s))\, ds. \tag{6.41}$$

Our next result immediately follows from Theorem 6.14, Lemmas 6.6 and 6.8, and the a priori estimates we have derived.

THEOREM 6.42. *If* $\{\tilde{S}_\varepsilon(t) \mid t \geqslant 0\}$ *is the semigroup associated with solutions to* (6.4a–e) *then for each* t, $\tilde{S}_\varepsilon(t)$ *is Fréchet differentiable on* Y_2 *at* U_0, *and its Fréchet derivative* $L(t, U_0)$ *satisfies the semilinear evolution equation*

$$\frac{d}{dt}(L(t, U_0)) = G_\varepsilon L(t, U_0) + \tilde{F}'_\varepsilon(\tilde{S}_\varepsilon(t)\, U_0)\, L(t, U_0),$$

$$L(0, u_0) = I \qquad \text{on } Y_2.$$

7. Hausdorff and Fractal Dimensions of the Attractors

In Section 4 we guaranteed the existence of a global attractor $\mathscr{A}_\varepsilon$ for the semigroup $\{S_\varepsilon(t) \mid t \geqslant 0\}$ associated with strong solutions of (4.1a–b), (4.2), (4.3a–c) in Y_1. In Section 5 we demonstrated that the attractor $\mathscr{A}_\varepsilon$ also lies in Y_2 and is an attractor for $\{S_\varepsilon(t) \mid t \geqslant 0\}$ in Y_2. If solutions to our system lie in the absorbing set for $\{S_\varepsilon(t) \mid t \geqslant 0\}$ then solutions have been shown to coincide with solutions of a system (6.4a–e) obtained by truncating the nonlinearities. Therefore we may compute the dimensionality of the attractor for $\{S_\varepsilon(t) \mid t \geqslant 0\}$ by computing the dimensionality of the attractor for

$\{\tilde{S}_\varepsilon(t) \mid t \geqslant 0\}$. We shall use the trace formula for Lyapunov exponents to estimate the Hausdorff and fractal dimensions of $\mathscr{A}_\varepsilon$. We find it convenient to introduce a change of variables by means of the isomorphism R^ε_γ: $Y_k \to Y_k$ ($k = 1$ or 2) which is defined below for $\varepsilon > 0$ and $\gamma > 0$:

$$R^\varepsilon_\gamma\colon U = \begin{pmatrix} \varphi \\ \psi \\ \theta \end{pmatrix} \to \tilde{U} = \begin{pmatrix} \varphi \\ \sqrt{\varepsilon}\,(\psi + \gamma\phi) \\ \theta \end{pmatrix}. \tag{7.1}$$

Applying R^ε_γ to the strong solution $U(t)$ of (6.4a–e) we have

$$\tilde{U}(t) = R^\varepsilon_\gamma U(t) = \begin{pmatrix} u(t) \\ v(t) = \sqrt{\varepsilon}\,(u_t(t) + \gamma u(t)) \\ w(t) \end{pmatrix}. \tag{7.2}$$

We hope that it shall not cause undue confusion that the triple $(u(t), v(t), w(t))$ shall for the remainder of this section apply to the transformed solution of (6.4a–e) rather than the original solution of (6.4a–e). Moreover, it is straightforward that the triple satisfies

$$\frac{\partial u}{\partial t} = \frac{1}{\sqrt{\varepsilon}} v - \gamma u,$$

$$\begin{aligned}
\frac{\partial v}{\partial t} &= \sqrt{\varepsilon}\, u_{tt} + \sqrt{\varepsilon}\, \gamma u_t = \sqrt{\varepsilon}\, u_{tt} + \sqrt{\varepsilon}\, \gamma \left(\frac{1}{\sqrt{\varepsilon}} v - \gamma u \right) \\
&= \sqrt{\varepsilon}\, u_{tt} + \gamma v - \sqrt{\varepsilon}\, \gamma^2 u \\
&= -\frac{1}{\sqrt{\varepsilon}} (1 + \varepsilon \tilde{f}_3(w)) \left(\frac{1}{\sqrt{\varepsilon}} v - \gamma u \right) + \frac{1}{\sqrt{\varepsilon}} u_{xx} \\
&\quad + \frac{1}{\sqrt{\varepsilon}} [\tilde{f}_2(w) - \tilde{f}_1(w)\, u] + \gamma u - \sqrt{\varepsilon}\, \gamma^2 u \\
&= \left(\gamma - \frac{1}{\varepsilon} - \tilde{f}_3(w) \right) (v - \sqrt{\varepsilon}\, \gamma u) + \frac{1}{\sqrt{\varepsilon}} [u_{xx} + \tilde{f}_2(w) - \tilde{f}_1(w)\, u].
\end{aligned} \tag{7.3}$$

Therefore we associate a new abstract Cauchy problem

$$\begin{aligned}
&\frac{d\tilde{U}(t)}{dt} = \tilde{G}_\varepsilon \tilde{U}(t) + \hat{F}_\varepsilon(\tilde{U}(t)), \qquad t > 0 \\
&\tilde{U}(0) = \tilde{U}_0 = (u_0, \sqrt{\varepsilon}\,(v_0 + \gamma u_0), w_0),
\end{aligned} \tag{7.4}$$

where the linear operator $\tilde{G}_\varepsilon\colon \mathscr{D}(G_\varepsilon) \to Y_k$, $(k=1, 2)$, is defined by

$$\tilde{G}_\varepsilon = \begin{pmatrix} -\gamma & \dfrac{1}{\sqrt{\varepsilon}} & 0 \\ -\dfrac{1}{\sqrt{\varepsilon}} A + \dfrac{\gamma}{\sqrt{\varepsilon}} - \sqrt{\varepsilon}\,\gamma^2 & \gamma - \dfrac{1}{\varepsilon} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the nonlinear mapping $\hat{F}_\varepsilon\colon Y_k \to Y_k$ is defined by

$$\hat{F}_\varepsilon(\tilde{U}) = \begin{pmatrix} 0 \\ \dfrac{1}{\sqrt{\varepsilon}} [\tilde{f}_2(w) - \tilde{f}_1(w) u] - \tilde{f}_3(w)(v - \sqrt{\varepsilon}\,\gamma u) \\ -\tilde{h}_1(u)\, w + \tilde{h}_2(u) \end{pmatrix}. \tag{7.6}$$

Similarly, we can show that the global solution of (7.4) exists and we denote the corresponding solution semigroup by $\hat{S}_\varepsilon(t)$, $t \geqslant 0$, i.e.

$$\hat{S}_\varepsilon(t)\, \tilde{U}_0 = \tilde{U}(t; \tilde{U}_0), \qquad t \geqslant 0. \tag{7.7}$$

We note that

$$\hat{S}_\varepsilon(t) = R_{-\gamma}^{1/\sqrt{\varepsilon}}\, \tilde{S}_\varepsilon(t)\, R_\gamma^{\sqrt{\varepsilon}}, \qquad t \geqslant 0. \tag{7.8}$$

Moreover, by equivalent arguments as in Sections 4 and 5 we can prove that for any ε, $0 < \varepsilon \leqslant \varepsilon_1$, there exists a global attractor for the equation (7.4), denoted by $\tilde{\mathscr{A}}_\varepsilon$, and $\tilde{A}_\varepsilon \subset Y_2$. It is easy to see that both mappings $R_\gamma^{\sqrt{\varepsilon}}$ and $R_{-\gamma}^{1/\sqrt{\varepsilon}}$ from Y_2 to itself are Lipschitz continuous. A direct application of Proposition 3.1 in Temam [44, p. 318] shows that the Hausdorff dimensions of the global attractors $\mathscr{A}_\varepsilon$ and $\tilde{\mathscr{A}}_\varepsilon$ are equal and the fractal dimensions of $\mathscr{A}_\varepsilon$ and $\tilde{\mathscr{A}}_\varepsilon$ are also equal. Therefore, it suffices to estimate the Hausdorff and fractal dimensions of the global attractor $\mathscr{A}_\varepsilon$ for the transformed equation (7.4).

Similarly, $\hat{S}_\varepsilon(t)\, \tilde{U}_0$ is Fréchet differentiable with respect to $\tilde{U}_0$ in Y_2, and its Fréchet derivative $\tilde{\Gamma}_\varepsilon(t, \tilde{U}_0) \in \mathscr{L}(Y_2)$ is given by

$$\tilde{\Gamma}_\varepsilon(t, \tilde{U}_0)\colon \tilde{V}_0 \in Y_2 \to \tilde{V}(t), \tag{7.9}$$

where $\tilde{V}(t)$, $t \geqslant 0$, is the mild solution of the linearized variational equation with the initial condition as follows:

$$\begin{aligned} &\frac{d\tilde{V}}{dt} = \tilde{G}_\varepsilon \tilde{V} + \tilde{L}_\varepsilon(\tilde{U}(t;\tilde{U}_0))\, \tilde{V}, \qquad t > 0 \\ &\tilde{V}(0) = \tilde{V}_0 \end{aligned} \tag{7.10}$$

where $\tilde{L}_\varepsilon(\tilde{U})$ is the Fréchet derivative of the mapping $\hat{F}_\varepsilon(\tilde{U})$ in Y_2, given by

$$\tilde{L}_\varepsilon(\tilde{U})\begin{pmatrix}\phi\\ \psi\\ \theta\end{pmatrix}=\begin{pmatrix}0\\ \dfrac{1}{\sqrt{\varepsilon}}[\tilde{f}_2'(w)\theta-\tilde{f}_1(w)\phi-\tilde{f}_1'(w)u\theta]\\ -\tilde{f}_3(w)[\psi-\sqrt{\varepsilon}\,\gamma\phi]-\tilde{f}_3'(w)\,\theta[v-\sqrt{\varepsilon}\,\gamma u]\\ -\tilde{h}_1(u)\theta-\tilde{h}_1'(u)w\phi+\tilde{h}_2(u)\phi\end{pmatrix}, \quad (7.11)$$

with $\tilde{U}=(u, v, w)$.

In order to use the Lyapunov exponent estimates to find an upper bound for the Hausdorff dimension $D_H(\tilde{A}_\varepsilon)$ (resp. the fractal dimension $D_F(\tilde{A}_\varepsilon)$), cf. Temam [44, Chap. 5 and 6], we consider m solutions $\tilde{V}_i(t)$ of (7.10), (corresponding to initial data $\tilde{V}_{0_i}\in Y_2$, $i=1, ..., m$.) By the trace formula,

$$\begin{aligned}&\|\tilde{V}_1(t)\wedge\cdots\wedge\tilde{V}_p(t)\|_{\Lambda^m Y_2}\\ &\qquad=\|\tilde{V}_{01}\wedge\cdots\wedge\tilde{V}_{0m}\|_{\Lambda^m Y_2}\exp\int_0^t \mathrm{Tr}[\tilde{\Pi}_\varepsilon(\tilde{S}_\varepsilon(\tau)\,\tilde{U}_0)\circ Q_m(\tau)]\,d\tau, \quad (7.12)\end{aligned}$$

where

$$\tilde{\Pi}_\varepsilon(\tilde{U})=\tilde{G}_\varepsilon+\tilde{L}_\varepsilon(\tilde{U}),$$

and $Q_m(t)$ is the orthogonal projection from Y_2 onto the subspace spanned by the m vectors $\{\tilde{V}_i(t): i=1, ..., m\}$.

Let $\Phi_i(t)=(\phi_i(t), \psi_i(t), \theta_i(t))$, $i=1, ..., m$, be an orthonormal basis of Range $Q_m(t)=\mathrm{span}[\tilde{V}_1(t), ..., \tilde{V}_m(t)]$, $t\geqslant 0$. Since each $\tilde{V}_i$ is a strongly continuous function, the basis functions $\Phi_i(t)$, $i=1, ..., m$, can also be chosen as strongly continuous functions of t.

Hence, we compute

$$\begin{aligned}\mathrm{Tr}[\tilde{\Pi}_\varepsilon(\tilde{U}(\tau))\circ Q_m(\tau)]&=\sum_{i=1}^m\langle\tilde{\Pi}_\varepsilon(\tilde{U}(\tau))\,\Phi_i(\tau), \Phi_i(\tau)\rangle_{Y_2}\\ &=\sum_{i=1}^m\{\langle\tilde{G}_\varepsilon\Phi_i(\tau), \Phi_i(\tau)\rangle_{Y_2}\\ &\quad+\langle\tilde{L}_\varepsilon(\tilde{U}(\tau))\,\Phi_i(\tau), \Phi_i(\tau)\rangle_{Y_2}\}. \quad (7.13)\end{aligned}$$

From (7.5) and (7.11), we can calculate

$$\begin{aligned}\langle\tilde{G}_\varepsilon\Phi_i(\tau), \Phi_i(\tau)\rangle_{Y_2}&=\left\langle-\gamma\phi_i+\frac{1}{\sqrt{\varepsilon}}\psi_i, \phi_i\right\rangle_{H^2}\\ &\quad+\left\langle\left(-\frac{1}{\sqrt{\varepsilon}}A+\frac{\gamma}{\sqrt{\varepsilon}}-\sqrt{\varepsilon}\,\gamma^2\right)\phi_i+\left(\gamma-\frac{1}{\varepsilon}\right)\psi_i, \psi_i\right\rangle_{H^1}\end{aligned}$$

$$= -\gamma(\|\phi_i\|^{(2)})^2 + \frac{1}{\sqrt{\varepsilon}}\langle \psi_i, \phi_i\rangle + \frac{1}{\sqrt{\varepsilon}}\langle \psi_{ix}, \phi_{ix}\rangle_{H^1}$$

$$-\frac{1}{\sqrt{\varepsilon}}\langle A\phi_i, \psi_i\rangle_{H^1} + \left(\frac{\gamma}{\sqrt{\varepsilon}} - \sqrt{\varepsilon}\,\gamma^2\right)\langle \phi_i, \psi_i\rangle_{H^1}$$

$$+\left(\gamma - \frac{1}{\varepsilon}\right)(\|\psi_i\|^{(1)})^2. \tag{7.14}$$

Since

$$-\frac{1}{\sqrt{\varepsilon}}\langle A\phi_i, \psi_i\rangle_{H^1} = -\frac{1}{\varepsilon}\langle A\phi_i, \psi_i\rangle - \frac{1}{\sqrt{\varepsilon}}\langle (A\phi_i)', \psi_i'\rangle$$

$$= -\frac{1}{\sqrt{\varepsilon}}\langle \phi_{ix}, \psi_{ix}\rangle - \frac{1}{\sqrt{\varepsilon}}\langle \phi_{ixx}, \psi_{ixx}\rangle$$

$$= -\frac{1}{\sqrt{\varepsilon}}\langle \phi_{ix}, \psi_{ix}\rangle_{H^1}, \tag{7.15}$$

thus

$$\langle \tilde{G}_\varepsilon \Phi_i(\tau), \Phi_i(\tau)\rangle_{Y_2} = -\gamma(\|\phi_i\|^{(2)})^2 + \frac{1}{\sqrt{\varepsilon}}\langle \phi_i, \psi_i\rangle$$

$$+\frac{1}{\sqrt{\varepsilon}}(\gamma - \varepsilon\gamma^2)\langle \phi_i, \psi_i\rangle_{H^1} + \left(\gamma - \frac{1}{\varepsilon}\right)(\|\psi_i\|^{(1)})^2$$

$$\leqslant -\gamma(\|\phi_i\|^{(2)})^2 + \frac{1}{\sqrt{\varepsilon}}|1 + \gamma - \varepsilon\gamma^2|\;\|\phi_i\|\;\|\psi_i\|$$

$$-\left(\frac{1}{\varepsilon} - \gamma\right)(\|\psi_i\|^{(1)})^2. \tag{7.16}$$

By HH(iv)′, (v), and (6.3a-b) there exists a constant $M > 0$ so that

$$\langle \tilde{L}_\varepsilon(\tilde{U}(\tau))\,\Phi_i(\tau), \Phi_i(\tau)\rangle_{Y_2}$$

$$= \left\langle \frac{1}{\sqrt{\varepsilon}}[\tilde{f}_2'(w)\,\theta_i - \tilde{f}_1(w)\,\phi_i - \tilde{f}_1(w)\,u\theta_i], \psi_i\right\rangle_{H^1}$$

$$-\langle \tilde{f}_3(w)(\psi_i - \sqrt{\varepsilon}\,\gamma\phi_i), \psi_i\rangle_{H^1} - \langle \tilde{f}_3'(w)\,\theta_i(v - \sqrt{\varepsilon}\,\gamma u), \psi_i\rangle_{H^1}$$

$$-\langle \tilde{h}_1(u)\,\theta_i + \tilde{h}_1'(u)\,w\phi_i - \tilde{h}_2'(u)\,\psi_i, \theta_i\rangle_{H^1}$$

$$\leqslant -\frac{1}{\sqrt{\varepsilon}}\langle \tilde{f}_1(w)\,\phi_i, \psi_i\rangle_{H^1} + \sqrt{\varepsilon}\,\gamma\langle \tilde{f}_3(w)\,\phi_i, \psi_i\rangle_{H^1}$$

$$-\langle \tilde{f}_3(w)\,\psi_i, \psi_i\rangle_{H^1}$$

$$-\langle \tilde{h}_1'(u)\, w\phi_i - \tilde{h}_2'(u)\, \phi_i, \theta_i \rangle_{H^1} - b(\|\theta_i\|^{(1)})^2$$

$$+\left\langle \left[\frac{1}{\sqrt{\varepsilon}} \tilde{f}_2'(w) - \frac{1}{\sqrt{\varepsilon}} \tilde{f}_1'(w)\, u - \tilde{f}_3'(w)(v - \sqrt{\varepsilon}\, \gamma u) \right] \theta_i, \psi_i \right\rangle_{H^1}$$

$$\leqslant c_1(\|\phi_i\|\ \|\psi_i\| + \|\phi_{ix}\|\ \|\psi_{ix}\|) + c_2(\|\phi_i\|\ \|\theta_i\| + \|\phi_{ix}\|\ \|\theta_{ix}\|)$$

$$+ c_3(\|\psi_i\|\ \|\theta_i\| + \|\psi_{ix}\|\ \|\theta_{ix}\|) + M(\|\psi_i\|^{(1)})^2 - b(\|\theta_i\|^{(1)})^2 \qquad (7.17)$$

where $c_1 = M(\varepsilon^{-1/2} + \gamma\varepsilon^{1/2})$. We use a priori estimates on w and w_x to produce a suitable c_2 which is independent of ε. Similarly, we can take $c_3 = (1/\sqrt{\varepsilon})\, \bar{C}_3$, where $\bar{C}_3$ is independent of ε.

Substituting the inequalities (7.16) and (7.17) into (7.13) we get

$$\operatorname{Tr}[\tilde{\Pi}_\varepsilon(\tilde{U}(\tau)) \circ Q_m(\tau)] \leqslant \sum_{i=1}^{m} -\gamma(\|\phi_i\|^{(2)})^2 - \left(\frac{1}{\varepsilon} - \gamma - M\right)(\|\psi_i\|^{(1)})^2$$

$$-b(\|\theta_i\|^{(2)})^2 + c_2(\|\phi_i\|\ \|\theta_i\| + \|\phi_{ix}\|\ \|\theta_{ix}\|)$$

$$+ c_3(\|\psi_i\|\ \|\theta_i\| + \|\psi_{ix}\|\ \|\theta_{ix}\|)$$

$$+ c_4(\|\phi_i\|\ \|\psi_i\| + \|\phi_{ix}\|\ \|\psi_{ix}\|), \qquad (7.18)$$

where $c_4 = c_1 + (1/\sqrt{\varepsilon})\, |1 + \gamma - \varepsilon\gamma^2|$. By Young's inequality,

$$c_4\, \|\phi_i\|\ \|\psi_i\| \leqslant \frac{\gamma}{4}\, \|\phi_i\|^2 + \frac{(c_4)^2}{\gamma}\, \|\psi_i\|^2, \qquad \text{and}$$

$$c_4\, \|\phi_{ix}\|\ \|\psi_{ix}\| \leqslant \frac{\gamma}{4}\, \|\phi_{ix}\|^2 + \frac{(c_4)^2}{\gamma}\, \|\psi_{ix}\|^2. \qquad (7.19)$$

Thus

$$c_4(\|\phi_i\|\ \|\psi_i\| + \|\phi_{ix}\|\ \|\psi_{ix}\|) \leqslant \frac{\gamma}{4}(\|\varphi_i\|^{(1)})^2 + \frac{(c_4)^2}{\gamma}(\|\psi_i\|^{(1)})^2$$

$$< \frac{\gamma}{4}(\|\phi_i\|^{(2)})^2 + \frac{(c_4)^2}{\gamma}(\|\psi_i\|^{(1)})^2. \qquad (7.20)$$

Similarly,

$$c_3(\|\psi_i\|\ \|\theta_i\| + \|\psi_{ix}\|\ \|\theta_{ix}\|) \leqslant \frac{b}{4}(\|\theta^{(i)}\|^{(1)})^2 + \frac{(c_3)^2}{b}(\|\psi_i\|^{(1)})^2, \qquad (7.21)$$

and

$$c_2(\|\phi_i\|\ \|\theta_i\| + \|\phi_{ix}\|\ \|\theta_{ix}\|) \leqslant \frac{b}{4}(\|\theta^{(i)}\|^{(1)})^2 + \frac{(c_2)^2}{b}(\|\phi_i\|^{(2)})^2. \qquad (7.22)$$

Substituting the inequalities (7.20)–(7.22) into (7.18) and using the fact that $\Phi_i(\tau)$ is an orthonormal basis in Y_2 we obtain

$$\begin{aligned}\operatorname{Tr}[\tilde{\Pi}_\varepsilon(\tilde{U}(\tau))\circ Q_m(\tau)] &\leqslant \sum_{i=1}^{m} -\left(\frac{3\gamma}{4}-\frac{(c_2)^2}{b}\right)(\|\phi_i\|^{(2)})^2 \\ &\quad -\frac{1}{\varepsilon}(\|\psi_i\|^{(1)})^2-\frac{b}{2}(\|\theta_i\|^{(1)})^2 \\ &\quad +\left(\gamma+M+\frac{(c_3)^2}{b}+\frac{(c_4)^2}{\gamma}\right)(\|\psi_i\|^{(1)})^2 \\ &\leqslant -c_0 m+\sum_{i=1}^{m} c_5(\varepsilon)(\|\psi_i\|^{(1)})^2, \end{aligned} \tag{7.23}$$

in which, since c_2 is independent of ε we can choose

$$\gamma=\frac{4(c_2)^2}{b}, \tag{7.24}$$

and from Section 5, $0<\varepsilon\leqslant\varepsilon_2$, so that

$$c_0=\min\left\{\frac{2(c_2)^2}{b}, \frac{1}{\varepsilon_2}, \frac{b}{2}\right\}, \tag{7.25}$$

which is a constant independent of ε, and

$$c_5(\varepsilon)=\frac{4(c_2)^2}{b}+M+\frac{(c_3(\varepsilon))^2}{b}+\frac{b(c_4(\varepsilon))^2}{4(c_2)^2}. \tag{7.26}$$

We can choose the orthonormal basis $\{\Phi_i(t), i=1, ..., m\}$ as

$$\Phi_i(t)=a_{i1}V_1(t)+\cdots+a_{im}(t)V_m(t), \qquad i=1, 2, ..., m, \tag{7.27}$$

where we have m^2 unknown coefficients. We have the constraints:

$$\|\Phi_i(t)\|_{Y_2}^2=\|a_{i1}V_1(t)+\cdots+a_{im}V_m(t)\|_{Y_2}^2=1, \qquad i=1, ..., m, \tag{7.28a}$$

and

$$\langle\Phi_i(t), \Phi_j(t)\rangle_{Y_2}=0, \qquad 1\leqslant i<j\leqslant m. \tag{7.28b}$$

If we choose the components $\psi_i(t)$ so that

$$(\|\psi_i\|^{(1)})^2=\alpha_i, \qquad i=1, ..., m, \tag{7.29}$$

with the given α_i's so that $0 < \alpha_i \leqslant 1$ and $\sum_{i=1}^{\infty} \alpha_i < \infty$, then we have a total of $(1/2)(m^2 + 3m)$ equality constraints. Since $m^2 > (1/2)(m^2 + 3m)$ if $m > 3$, we can determine the m^2 unknown coefficients in (7.27) to satisfy all the above constraints. Therefore we choose $\psi_i(t)$ so that

$$(\|\psi_i(t)\|^{(1)})^2 \leqslant \alpha_i, \qquad t \geqslant 0, \quad i = 1, ..., m. \tag{7.30}$$

This modification does not affect any of the steps starting from (7.14). As a result, we obtain from (7.23) that

$$\begin{aligned} q_m &= \limsup_{t \to \infty} \sup_{\tilde{U}_0 \in \mathscr{A}_\varepsilon} \sup_{\substack{\tilde{V}_{0i} \in Y_2 \\ \|\tilde{V}_{0i}\|_{Y2} \leqslant 1 \\ i=1,...,m}} \left\{ \frac{1}{t} \int_0^t \operatorname{Tr}[\tilde{\Pi}_\varepsilon(\hat{S}_\varepsilon(\tau)\, \tilde{U}_0) \circ Q_m(\tau)]\, d\tau \right\} \\ &\leqslant -c_0 m + c_5(\varepsilon) \sum_{i=1}^{m} \alpha_i \\ &\leqslant -c_0 m + c_6(\varepsilon), \end{aligned} \tag{7.31}$$

where

$$c_6(\varepsilon) = c_5(\varepsilon) \sum_{i=1}^{\infty} \alpha_i. \tag{7.32}$$

Therefore, the Lyapunov exponents $\{\mu_i\}$ satisfy

$$\mu_1 + \cdots + \mu_m \leqslant q_m \leqslant -c_0 m + c_6(\varepsilon), \qquad \forall m \in \mathbb{N}, \quad m > 3. \tag{7.33}$$

Using the theory of Hausdorff and fractal dimensions of global attractors, cf. [44, Theorem V.33], we can conclude that for a suitably large integer m such that $m > 3$ and

$$\frac{c_6(\varepsilon)}{c_0} < m \leqslant \frac{c_6(\varepsilon)}{c_0} + 1, \tag{7.34}$$

it follows that $\mu_1 + \cdots + \mu_m < 0$. Consequently, we have the following result.

THEOREM 7.35. *If $\varepsilon \in (0, \varepsilon_2]$, there exists $m > 0$ so that for the global attractor $\mathscr{A}_\varepsilon$ of the semiflow defined by* (6.4a–e) *in Y_2,*

$$D_H(\mathscr{A}_\varepsilon) \leqslant m \qquad \textit{and} \qquad D_f(\mathscr{A}_\varepsilon) \leqslant 2m,$$

where $D_H(\mathscr{A}_\varepsilon)$ and $D_f(\mathscr{A}_\varepsilon)$ denote the Hausdorff and fractal dimensions of $\mathscr{A}_\varepsilon$ respectively.

Remark. From the description of the constants in (7.17), (7.26) and (7.32), we see that $m \sim c/\varepsilon$, where c is a constant, as $\varepsilon \to 0^+$.

8. Uppersemicontinuity of the Attractors

We recall that $\mathscr{A}_0$ represents the natural embedding of $\mathscr{A}$ (the attractor for the standard parabolic Hodgkin–Huxley system) into three component space Y_1. We shall establish that $\lim_{\varepsilon \downarrow 0} \delta_{Y_1}(\mathscr{A}_\varepsilon, \mathscr{A}_0) = 0$ and use this to conclude that in essence the long term dynamics of the singularly perturbed problem converge to those of the standard system. This in turn validates the setting of the small parameter ε to 0. As previously stated, a scalar variant of this type of result was obtained by Hale and Raugel [23, 24] for a singularly perturbed equation of the form

$$\varepsilon u_{tt} + u_t - \Delta u = f(u) - g,$$

with homogeneous Dirichlet boundary data, in $\Omega \subseteq \mathbb{R}^n$, $n \leqslant 3$. Methods similar to those of Hale and Raugel [23, 24] were applied by Debussche [8] to describe the convergence of attractors for the singularly perturbed Cahn–Hilliard equation

$$\varepsilon u_{tt} + u_t + \mu \Delta^2 u = \Delta f(u).$$

We note here also the extensive work of Babin and Vishik (cf. [2], and the references therein) on semicontinuity of attractors, including the convergence of attractors of hyperbolic equations to attractors of parabolic equations.

Our final result is

Theorem 8.1. *If $\mathscr{A}_\varepsilon$ denotes the attractor for the singularly perturbed Hodgkin–Huxley system of Section* 4, *and $\mathscr{A}_0$ denotes the embedding of the global attractor for the standard Hodgkin–Huxley system in Y_1 defined by* (3.19), *then*

$$\lim_{\varepsilon \downarrow 0} \delta_{Y_1}(\mathscr{A}_\varepsilon, \mathscr{A}_0) = 0. \tag{8.2}$$

Proof. We proceed in a manner similar to that of Hale and Raugel [24]. By virtue of Corollaries 5.68, 5.69 and the known properties of the attractor $\mathscr{A}$ there exists a bounded set $B_1 \subset Y_2$ such that

$$\left(\bigcup_{0 < \varepsilon \leqslant \varepsilon_2} \mathscr{A}_\varepsilon \right) \cup \mathscr{A}_0 \subseteq B_1, \tag{8.3}$$

and there exists a positive constant k_2 so that for any trajectory of solutions of (4.1a–b), (4.2), (4.3a–c), $\{(u_\varepsilon(t), u_{\varepsilon_t}(t), w_\varepsilon(t)) \mid t \geqslant 0\} \subseteq \mathscr{A}_\varepsilon$, we have

$$\sup_{t \geqslant 0} \sqrt{\varepsilon}\, \|\partial^2 u_\varepsilon(t)/\partial t^2\| \leqslant k_2. \tag{8.4}$$

If we refer to Theorem 1.1 of Hale and Raugel [24], we see that we will establish the desired result if we can demonstrate that for every sequence $\{\varepsilon_n\} \downarrow 0$ with corresponding trajectory of solutions $(u_n(t), u_n'(t), w_n(t)) \subseteq \mathscr{A}_{\varepsilon_n}$, there is a subsequence $\{\varepsilon_{j_n}\}$ such that

$$\lim_{n \to \infty} (u_{j_n}(0), u_{j_n}'(0), w_{j_n}(0)) = (\bar{u}_0, \bar{v}_0, \bar{w}_0) \in Y_1 \tag{8.5a}$$

with

$$(\bar{u}_0, \bar{v}_0, \bar{w}_0) \in \mathscr{A}_0. \tag{8.5b}$$

Let $(u_n(t), u_n'(t), w_n(t)) \in \mathscr{A}_{\varepsilon_n}$, $t \geqslant 0$. By (8.3), $\bigcup_{t \geqslant 0} \bigcup_{n \in N} \{u_n(t), w_n(t)\}$ is a pre-compact set in $X_1 = H^1(0,1) \times L_2(0,1)$, and the family of mappings $\{u_n(t), w_n(t)\} \in C(R^+; X_1)$, $n \geqslant 0$, is equicontinuous from $[0, \infty)$ into X_1. Let J_m, $m \geqslant 0$, be a sequence of compact intervals of $[0, \infty)$ such that $J_m \subset J_{m+1}$, $m \geqslant 0$, and $\bigcup_{m \in N} J_m = [0, \infty)$. By Ascoli's theorem, there exists a subsequence $\{u_{n_0}, w_{n_0}\}$ of $\{u_n, w_n\}$ such that $\{u_{n_0}, w_{n_0}\}$ converges to $(\bar{u}, \bar{w})$ in $C(J_0; X_1)$, and using Ascoli's theorem again, one shows by induction that there is a subsequence $\{u_{n_{m+1}}, w_{n_{m+1}}\}$ of $\{u_{n_m}, w_{n_m}\}$ such that $\{u_{n_{m+1}}, w_{n_{m+1}}\}$ converges to $(\bar{u}, \bar{w})$ in $C(J_{m+1}; X_1)$. Finally taking a diagonal subsequence in the usual way, there exists a subsequence of positive numbers ε_{j_n} of ε_n and the corresponding subsequence $\{u_{j_n}, w_{j_n}\}$ of $\{u_n, w_n\}$, such that

$$\{u_{j_n}, w_{j_n}\} \to (\bar{u}, \bar{w}) \tag{8.6}$$

in $C(J; X_1)$ for any compact interval $J \subset [0, \infty)$, and $(\bar{u}, \bar{w})$ belongs to $C(R^+; X_1)$. Furthermore, due to (8.3), we are assured of $k_1 > 0$ so that

$$\sup_{t \geqslant 0} \{\|\bar{u}(t)\|^{(1)}, \|\bar{w}(t)\|\} \leqslant k_1. \tag{8.7}$$

That is,

$$(\bar{u}, \bar{w}) \in C_b(R^+; X_1) \equiv \{v \in C(R^+; X_1) \mid \sup_{t \in R^+} \|v(t)\|_{X_1} \text{ is bounded}\}.$$

Moreover, from (8.4), it follows that

$$\left(\sup_{j \geqslant 0} \varepsilon_{j_n} \left\|\frac{\partial^2 u_{j_n}(t)}{\partial t^2}\right\|\right) \to 0 \qquad \text{as} \quad j_n \to +\infty. \tag{8.8}$$

On the one hand, $\partial u_{j_n}/\partial t$ converges in $\mathscr{D}'(I; H^1(0,1))$ (that is, in the sense of distributions) to $\partial \bar{u}/\partial t$, for any bounded open interval $I \subset R^+$. On the other hand, since

$$\frac{\partial u_{j_n}}{\partial t} = \left(\frac{1}{1+\varepsilon_{j_n} f_3(w_{j_n})}\right)\left[-\varepsilon_{j_n}\frac{\partial^2 u_{j_n}}{\partial t^2} - Au_{j_n} - f_1(w_{j_n})\, u_{j_n} + f_2(w_{j_n})\right],$$

(8.6) and (8.8) imply that $\partial u_{j_n}/\partial t$ converges in $C(J; (H^1_N(0,1))')$ to $(A\bar{u} - f_1(\bar{w})\,\bar{u} + f_2(\bar{w}))$ for any compact interval $J \subset R^+$. Here $(H^1_N(0,1))'$ is the dual space of $H^1(0,1)$ with the Neumann boundary conditions. By the uniqueness of the limit in $\mathscr{D}'(I; (H^1_N(0,1))')$, cf. Kato [30], we have

$$\frac{\partial \bar{u}}{\partial t} = -A\bar{u} - f_1(\bar{w})\,\bar{u} + f_2(\bar{w}). \tag{8.9}$$

By (8.7), we deduce from (8.9) that $\partial \bar{u}/\partial t$ belongs to $C_b(R^+; (H^1_N(0,1))')$. Since, by (8.3), $\sup_{t \in R^+} \|\partial u_{j_n}(t)/\partial t\|^{(1)}$ remains bounded when j_n tends to infinity, the convergence of $\partial u_{j_n}/\partial t$ to $\partial \bar{u}/\partial t$ in $C(J; (H^1_N(0,1))')$, for any compact set $J \subset R^+$, implies the convergence of $\partial u_{j_n}/\partial t$ to $\partial \bar{u}/\partial t$ in $C(J; L_2(0,1))$ for any compact set $J \subset R^+$; moreover, $\partial \bar{u}/\partial t$ belongs to $C_b(R^+; L_2(0,1))$. Thus $\partial \bar{u}/\partial t + f_1(\bar{w})\,\bar{u} - f_2(\bar{w})$ belongs to $C_b(R^+; L_2(0,1))$. From the equality

$$-A\bar{u} = \frac{\partial \bar{u}}{\partial t} + f_1(\bar{w})\, u - f_2(\bar{w}), \tag{8.10}$$

and the regularity properties of the operator A, we can conclude that $\bar{u}$ belongs to $L_\infty(R^+; H^2(0,1))$. Finally, we have proved that $\bar{u}$ belongs to $L_\infty(R^+; H^2(0,1)) \cap W^{1,\infty}(R^+; L_2(0,1))$.

Since $\lim_{n\to\infty}(-h_1(u_{j_n})\, w_{j_n} + h_2(u_{j_n})) = -h_1(\bar{u})\,\bar{w} + h_2(\bar{u})$ in $C(J; L_2(0,1))$, and $\partial w_{j_n}/\partial t = -h_1(u_{j_n})\, w_{j_n} + h_2(u_{j_n})$, this implies that $\lim_{n\to\infty} \partial w_{j_n}/\partial t = \partial \bar{w}/\partial t = -h_1(\bar{u})\,\bar{w} + h_2(\bar{u})$ in $C(J; L_2(0,1))$ for any compact interval $J \subset R^+$. Furthermore, $\bar{w} \in L_\infty(R^+; L_2(0,1)) \cap W^{1,\infty}(R^+; L_2(0,1))$. We see that $(\bar{u}, \bar{w})$ satisfies:

$$\bar{u}_t + A\bar{u} + f_1(\bar{w})\,\bar{u} - f_2(\bar{w}) = 0 \tag{8.11}$$

$$\frac{\partial \bar{w}}{\partial t} = -h_1(\bar{u})\,\bar{w} + h_2(\bar{u}). \tag{8.12}$$

Therefore, by the definition of $\mathscr{A}$ and $\mathscr{A}_0$, we see that $(\bar{u}(t), \bar{u}'(t), \bar{w}(t))$ belongs to $\mathscr{A}_0$ for any $t \geqslant 0$. Because $(u_{j_n}, \partial u_{j_n}/\partial t, w_{j_n})$ converges to $(\bar{u}, \partial \bar{u}/\partial t, \bar{w})$ in $C(J; Y_1)$ for any compact interval J of R^+, $(u_{j_n}(0), \partial u_{j_n}(0)/\partial t, w_{j_n}(0))$ in particular converges to $(\bar{u}(0), \partial \bar{u}(0)/\partial t, \bar{w}(0)) \in \mathscr{A}_0$ in Y_1.

We have thereby shown that the Hale–Raugel criterion is met and we have consequently established our result.

In closing, the authors feel that questions pertaining to the convergence of hyperbolic singular perturbations of parabolic systems are important in a larger context. It can be argued that a telegrapher's equation provides a more rigorous description of physical diffusion or biological dispersion than the traditional heat equation, cf. [17] for references. The authors therefore feel that the qualitative investigation of systems of equations which involve hyperbolic equations as singular perturbations of reaction diffusion equations is an important area for further studies.

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